SPECTRAL ANALYSIS AND THE STUDY OF INDIVIDUAL DIFFERENCES IN THE PERFORMANCE OF ROUTINE, REPETITIVE TASKS

A Technical Report
prepared by

ROBERT P. ABELSON

Project Designation NR 150-088
Office of Naval Research Contract N6onr-27020
with
PRINCETON UNIVERSITY
Princeton, New Jersey

Principal Investigator: Harold Gulliksen, Psychology Department

EDUCATIONAL TESTING SERVICE
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I

INTRODUCTION

This dissertation is concerned with individual differences in certain characteristics of the performance of routine, repetitive tasks. Such tasks require the production of well-learned responses over a long period of time to a relatively constant stimulus situation.

The present research employed an experimental task as a paradigm for a repetitive, routine task. The hypotheses under which this research was carried out were 1) that reliable individual differences in the performance of repetitive tasks can be discovered by objective analysis of long series of experimental observations which are ordered along the time dimension, and 2) that the differences discovered by objective analysis in one task situation are of a general nature: i.e., certain stable features of personality are manifested on all repetitive, routine tasks.

Both of these hypotheses require amplification. Firstly, the fact that long series of observations ordered along the time dimension are being analyzed is of great importance. Time-ordering as a property of psychological data has never been adequately treated in the psychological literature (see Section II). The only statistical measures of long series of observations which have been used extensively are the mean and the standard deviation. In studies of learning or fatigue, the chief interest centers upon <u>changes in the mean level</u> of perform-

ance, and, to a lesser degree, upon changes in the standard deviation. Measures of the strictly serial characteristics of the data in these studies are of less moment, since most of the relevant information about the process is conveyed by the changes in these two measures. But now consider a set of time-ordered data in which there are no major changes from the beginning of the series to the end of the series, a set of data as might arise in studies of attention, psychophysical judgments, work without decrement, or various performance tasks. Here it is clear that the mean and the standard deviation, though they are important to the analysis of the data, do not exhaust the relevant information conveyed by the data. There is residual information concerning the serial dependencies of observations upon observations prior in time--what might loosely be called the rhythmical properties of the data. It is these properties which are a main concern of this paper. The mathematical analysis of these properties will constitute a major objective of study.

A long detour is taken in Sections II, III, IV and V to treat the relevant historical and mathematical material, before returning in Section VI to the experimental study of individual differences in the performance of routine, repetitive tasks.

What features of personality are manifest in the performance of such tasks? In the absence of a systematic, theoretical framework to guide us (since we are not primarily concerned with the extensively explored areas of learning fatigue or motor skill), we will be content here with a list of concepts to which these manifested features of personality might be allied.

1. Rigidity
2. Level of aspiration
3. Involvement
4. Concern over errors
5. Stability
6. Attention

The conceptual framework will be sharpened and elaborated in later sections.

II

HISTORY OF THE ANALYSIS OF TIME-ORDERED SERIES IN PSYCHOLOGICAL DATA[1]

Time-ordered series have appeared in psychological data since the inception of experimental psychology. However, their mathematical and statistical properties have almost always been ignored or taken lightly, with the result that the analysis of such data has been arbitrary, spotty, and erroneous.

Two of the oldest lines of research deal with attention waves and with work rhythms. Profuse references to these researches may be found in Guilford (19) and Bills (5), respectively. The research prior to 1900 did not recognize time-ordering as a salient property of its data. In 1905, however, Seashore and Kent (31) published a lengthy article on the subject of periodic change in continuous mental work. This article contained a great deal of data plotted in serial order. The "mental work" involved attending to a tone which oscillated systematically in intensity about the auditory threshold, and the data consisted of the intensities of the tone at those times when it became just audible (or inaudible). Seashore and Kent claim that periodicities of three different wave lengths are apparent upon inspection of the data: second-waves, minute-waves, and hour-waves. These three varieties of waves are then reified and discussed. But no objective evidence for their existence is presented. Indeed, it is doubtful that an independent observer inspecting the same data records would have postulated the same three waves.

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[1]Data arising from learning or fatigue studies are not considered, since the rhythmical properties of such data are overshadowed in importance by changes in the mean level of performance.

It must have been apparent to many experimenters that mere inspection of time-ordered data was unsatisfactory as an analytical technique. Sarvis (30), in particular, decried the lack of analytical tools and attempted some objectification of his own data. The task he set for his subjects was that of tracing a square repeatedly without pausing. He timed the completion of each circuit around the square and plotted these times in serial order. His criteria for rhythmicality were the distances between successive troughs, and a more or less intuitive evaluation of the character of the intervening succession of points. If the inter-peak and inter-trough distances seemed relatively constant and the over-all wave generally smooth, a rhythm was said to have been established; but it was very rare for an empirical record to have these properties. As Sarvis himself pointed out, a combination of different rhythms could make the wave picture quite complicated. Here Sarvis resorted to intuition, with the statement that years of study are required to train the eye to the detection of hidden periodicities in graphical records.

Another who felt that objective methods were meager was Philpott (27, 28). His approach took a very different turn. He noticed that when work curves from many different tasks and experimental sources were summed together, the resulting composite curve showed wide oscillations. Claiming that these oscillations were too large to be accounted for by chance, he postulated that work curves on all tasks and in all situations conform to what he called a "master time-table." Assuming (1) that all rhythms in the work curve must have periodicities equal to <u>integers</u> on the time scale and should occur with equal probability, and (2) that the work curve must be plotted against the logarithm of time instead of against time itself, Philpott studied only the troughs in his composite work curve

and devised a scheme which he thought would describe the inherent organization of the curve itself. While he himself has not tested his scheme against independent experimental data, several of his students (cf. Chen (7)) have done so. The criterion of conformity of the model to the data is, unfortunately, subjective. It consists of a judgment as to whether the troughs in the empirical work curve fall near certain positions on the time axis. An evaluation of Philpott's scheme would have to condemn it as grossly implausible; his assumptions (especially that of a master time-table) are gratuitous, his mathematical treatment is naive, and his empirical validation is unconvincing.

The difficulty with many of these attempts is that they have taken for granted the idea that curves of work or fluctuations of attention should follow the pattern of a small number of well-defined sinusoidal components. Rhythm has been unjustifiably bound to variability. This idea undoubtedly stems from the feeling that since many physiological phenomena are almost exactly rhythmical, psychological phenomena should be so too, and should correlate with physiological rhythms such as heart beat, breathing, the Traube-Hering wave of blood pressure, or some central neurological rhythm--a "scanning wave" in the brain (36). Often strict rhythmicality is postulated for psychological phenomena which are in fact not rhythmical, but random. A case in point is Bills' study of "mental blocking," which is treated in detail in Section V.

An entirely different line of research covers those time-ordered series where the natural expectation is one of randomness and not rhythmicality. These series are characterized by the fact that the variable under measurement is ordinarily not quantitative, but qualitative. There

are psychophysical situations known to many authors (e.g., Flynn (17), Arons and Irwin (2), Preston (29) and Day (10)) in which the presence of non-random serial patterns of response have been detected. Similarly, non-random patterning has been noticed by Thorndike (37) in verbal responses. Goodfellow (18) and Skinner (33) have observed the tendency to avoid repetition of response categories in a popularized experiment on mental telepathy. Miller and Frick (25) have discussed certain aspects of sequential patterns of responses in multiple-choice situations. In some of these cases, the mathematical treatment of time-ordered series has been rather sophisticated, particularly with Day, who used autocorrelational analysis, and Miller and Frick, who discussed the Markoff process. However, neither of these methods is quite general enough, although autocorrelational analysis comes close, and deserves detailed consideration (Section III).

A third line of research which departs somewhat from the psychological in order to embrace engineering concepts is that of "operational analysis" (12). This field has roots in both military psychology and in cybernetics (42). Operational analysis deals with electrical and mechanical systems in which a human operator intervenes in one or more parts of the system. The activity of the operator usually consists of a long series of ongoing motor responses to a given perceptual situation, and is thus very similar to the activity which is the subject of experimental investigation in this dissertation. The prototype of this perceptual-motor activity as treated by operational analysis is pursuit tracking (15). In pursuit tracking the operator, by manual manipulation, attempts to maintain a pointer or cross-hair in correspondence with a target which moves in the visual field. Two pursuit tracking studies

which have employed mathematical methods of time-series analysis are those of Flood (14) and of Krendel (24). Flood made use of autocorrelational analysis and discovered a coherent organization to the time-series properties of his data. Krendel, on the other hand, employed "spectral analysis" to describe his data. Spectral analysis is the method eventually chosen here as being ideal for the general analysis of time-ordered data and in particular for the analysis of individual differences on a repetitive perceptual-motor task. It must be emphasized that while this paper employs both a similar task and a similar analytical method to those of operational analysis, its outlook is essentially antithetical to that of operational analysis. The present emphasis is upon the individual as a personal organism whose activities require explanation in and of themselves, and not as an operator in a mechanical and electrical system.

In summary, investigations of psychological phenomena wherein the experimental variable undergoes fluctuations with time have suffered from lack of analytic methods to deal with the time-ordered properties of the data. Early methods were characterized by an appeal to inspection of the data and by arbitrary, subjective statements about rhythms which were presumed to exist in graphical records. Later emphasis was more objective, and a small number of recent studies have employed methods powerful enough to cope with the analysis of the complicated properties of time-ordered data.

III

METHODS OF TIME-SERIES ANALYSIS

Possible methods of analysis of time-ordered observations may be culled from the literature of the other sciences. Although a great number of methods exist, the majority of them can be subsumed within the three main categories: Quality control, autocorrelational analysis, and spectral analysis. The first two will be discussed in this section, and spectral analysis in Section IV.

Quality Control

The methods of quality control, developed originally by Shewhart (32) for industrial applications, deal with the class of time-series arising when measurements are made on each of many successive machine products. To be more precise, measurements are made or samples drawn at regular intervals from the output population of a machine in operation. The purpose of quality control is the detection and correction of improper machine operation. There are many typical ways in which a machine can get "out of control" and produce products which violate, in one or more ways, the manufacturer's standards of quality. To most of these types of defections there corresponds an optimal statistical test carried out upon the sampled products. These tests are designed to serve as ready danger-signals. If a machine goes out of control, the interest is not in analyzing in their own right the interesting and peculiar properties of the resulting series of measurements, but in making the indicated machine corrections as quickly as possible.

A machine is said to be "in control" when firstly, measurements on successive products are independent. This independence of successive events is what we mean when we say that a time-process is <u>random</u>. The further requirement usually imposed is that the individual measurements be normally distributed with some fixed, predetermined variance. Olmstead (26) has enumerated the typical ways in which a machine product can be out of control. (The statistical tests he gives for the detection of each type do not interest us here.) These are:

1. A gross error or "blunder"; i.e., an individual product appears which is very much out of line with the over-all distribution.

2. A shift in the mean measurement of a group of products.

3. A shift in the variance of measurement of a group of products.

4. A gradual change (trend) in the mean measurement.

5. A regular pattern of change (cycle) in the mean measurement.

Now, Shewhart has concluded from his own private observations that it is virtually impossible to find a series of repetitive human performances which is "in control." If this be true, it would be of great interest to explain this curious finding, perhaps describe the nature of the usual out-of-controlness, and most interesting of all, to attempt to find meaningful individual differences on the basis of degree and type of out-of-controlness. This approach might be entitled "Human Quality Control." It was a discussion of this possibility between Dr. Shewhart and Professor Harold Gulliksen that gave impetus to the present project.

That the quality control method is not of general experimental applicability can be held on three grounds:

1. Quality control analysis is a set of diverse, particularized procedures which do not lend themselves to a large study of a group of

individuals. The analysis is too diffuse. One individual may show broken trends, another a non-normal distribution, two more a predilection for "blunders," etc. There seems no a priori way to organize such findings.

2. The interest in quality control is primarily with correction and not with description and explanation. If, say, a changing variability is encountered, we would want to know "Why?" and "In what way?" Quality control analysis probably will not tell us this. The experience gained in answering these questions for machine operation is not likely to prove useful when dealing with human subjects.

3. It is experimentally difficult to produce human performances in as great profusion as machine products can be produced, so that any aberrances which occurred with rare frequency or over very long intervals would go undiscovered.

It is still possible that for certain very particular psychological hypotheses, quality control methods would prove useful. But they do not serve as a general procedure for handling time-ordered psychological data, nor for the experimental study at hand. It is well to add that the same arguments apply to sequential analysis (35, 40), an outgrowth and refinement of quality control. The observation that human serial performances are almost never random is not being ignored. This human propensity toward out-of-controlness can be uncovered by other methods which will prove of more general applicability than quality control analysis as far as psychological data is concerned.

Autocorrelational Analysis

In contrast to the quality control method, autocorrelational analysis is of quite general applicability to large classes of time-ordered

data. The method has been applied to meteorological and economic time-series data with fair success; also, as has been mentioned, Day (10) and Flood (14) employed it with psychological data. The mathematical background of autocorrelational analysis is given in Kendall (22); historical and mathematical material appears in Davis (9).

In order to consider autocorrelational analysis here, it will be necessary to define a time series <u>population</u> and a <u>stationary</u> time-series. A <u>sample</u> of time-ordered observations will be denoted by $(X_1 \; X_2 \; X_3 \; \ldots\ldots \; X_t \; \ldots\ldots X_{N-1} \; X_N)$, or more simply by $(X_t)_{t=1,2\ldots N}$, where X is the variable under consideration and the subscript denotes the time-order of occurrence[2], t being a general subscript denoting the t:th observation, and N being the total number of observations. Now in the same fashion that a single observation is considered to be a sample from a statistical population of potentially occurring single observations, the set $(X_t)_{t=1,2\ldots N}$ must be considered to be a sample from a population of potentially occurring sets of N time-ordered observations each. This population is N-dimensional, and will be denoted[3] by $\{X_t\}_{t=1,2\ldots\ldots N}$. The objective of any method of time-series analysis is to induce the nature of the population $\{X_t\}_{t=1,2\ldots N}$ from one or more samples $(X_t)_{t=1,2\ldots N}$.

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[2] If the observations are made at equal time intervals, the time-order of occurrence is equivalent to the actual time of occurrence, where the first observation occurs at time t=1 and the interval between successive occurrences is taken as the unit of time.

[3] Where reference is made to the population of X's at some fixed, single value of t instead of the population of X's at all values of t, the identification t=1,2... will be dropped from the notation.

In most cases, only one sample $(X_t)_{t=1,2...N}$ is available. It would be hopeless to induce anything about a population from a sample of one were it not for the assumption of stationarity. Consider the sample to be translated along the time axis a distance of K units (i.e., K new observations are added at the end of the series and K observations at the beginning are dropped). Compare the associated population $\{X_t\}_{t=K+1,K+2,...K+N}$ with the original population $\{X_t\}_{t=1,2...N}$. If these two populations are identical--indistinguishable from one another--for any positive or negative integer value of K, the time-series is said to be stationary. A stationary series is characterized by the fact that the data we find now is equivalent (in a sampling sense) to what we might have found before or might find later. In other words, time has no effect as such, but only insofar as it orders the observations with respect to each other. The power of this assumption will become apparent as we progress.

Autocorrelational analysis approaches the population $\{X_t\}_{t=1,2...N}$ with the question, "Are $X_1, X_2, ...X_N$ independent, and if not, what is the extent of the dependence of any observation on those preceding it?" In other words, we inquire into the N-dimensional joint probability distribution of $X_1, X_2, ...X_N$. If all the X's are independent, then the series is random, and the joint probability distribution is simply equal to the product of the separate probability distributions $\{X_1\}$, $\{X_2\}, ...\{X_N\}$. Further, for the stationary case, these separate distributions are identical.

Now, in the case where the X's are not all independent, we ask, "How can we best summarize the dependencies?" If attention is restricted to linear dependencies and stationary time-series, the question is answered

as follows: Consider the linear regression of $\{X_2\}$ on $\{X_1\}$. Continued sampling from the population $\{X_1, X_2\}$ will produce a bivariate surface with a linear regression and a certain correlation coefficient. Since time has no effect as such, but only insofar as it orders the observations, the regression of $\{X_2\}$ on $\{X_1\}$ is equal to the regression of $\{X_3\}$ on $\{X_2\}$, and, in general, of $\{X_{K+1}\}$ on $\{X_K\}$. Thus we may think of a general regression effect of an observation on the observation prior to it, the so-called autocorrelation of lag one, or first-lag correlation, R_1 . Similarly, there is a general regression effect of $\{X_{K+2}\}$ on $\{X_K\}$, the second-lag correlation, R_2, --the correlation between an observation and the observation prior to it save one.

In general, there is a regression effect of $\{X_{K+j}\}$ on $\{X_K\}$, the j:th - lag correlation, R_j. Computationally, R_j conforms to the ordinary correlation form, the covariance divided by the product of the standard deviations. In this case, however, the standard deviation of the observations X_1 through X_{N-j} is very nearly equal to the standard deviation of the observations X_{j+1} through X_N, and both may be taken as equal to the standard deviation of the observations X_1 through X_N .
Thus

$$R_j = {}^{Q_j}/_{Q_o} \qquad \text{, where} \tag{3.1}$$

$$Q_j = \frac{\sum\limits_{t=1}^{N-j} X_t X_{t+j}}{N-j} - \frac{\left[\sum\limits_{t=1}^{N-j} X_t\right]\left[\sum\limits_{t=1}^{N-j} X_{t+j}\right]}{(N-j)^2} \tag{3.2}$$

(Q_j is called the j:th lag covariance.)

$$(3.3) \qquad Q_o = \frac{\sum_{t=1}^{N} X_t^2}{N} - \frac{\left[\sum_{t=1}^{N} X_t\right]^2}{N^2} = \sigma_x^2$$

All the linear dependencies in a series of N observations can be expressed by the lag correlations $R_1, R_2, \ldots R_{N-1}$. For a random series, all $R_j = 0$. If the R_j tend toward alternation of positive and negative values, the time series is non-random in that neighboring observations tend to be unlike one another; the series shows sharp quick ups and downs. If the R_j are high and positive for small values of j, trailing off to zero or negative values as j is increased, the time series is non-random in that observations tend to be like their immediate neighbors but unlike observations farther removed; the series shows a slow, gradual, wandering trend or cycle.

In Figure 1 are presented three time-ordered series of empirical observations (from the present experimental study). Series A is almost random; Series B shows quick ups and downs; Series C shows slow, cyclical variations. In Figure 2 are presented the scatterplots for the regression of $\{X_{K+1}\}$ on $\{X_K\}$, the autocorrelations of lag one, for each of the empirical series. For Series A, the scatterplot is circular, indicating no correlation; for Series B there is evidently a negative correlation, and for Series C, a positive correlation. In Figure 3, the first 20 lag autocorrelations for each of these series are shown; these plots conform in general to the descriptions of the previous paragraph.

The lag correlations R_j are constrained in relation to each other; that is, they are not independent. For instance, if R_1 is very close to 1.00, R_2 must be quite high also, since a lag of two observations is

FIGURE 1. SERIES A

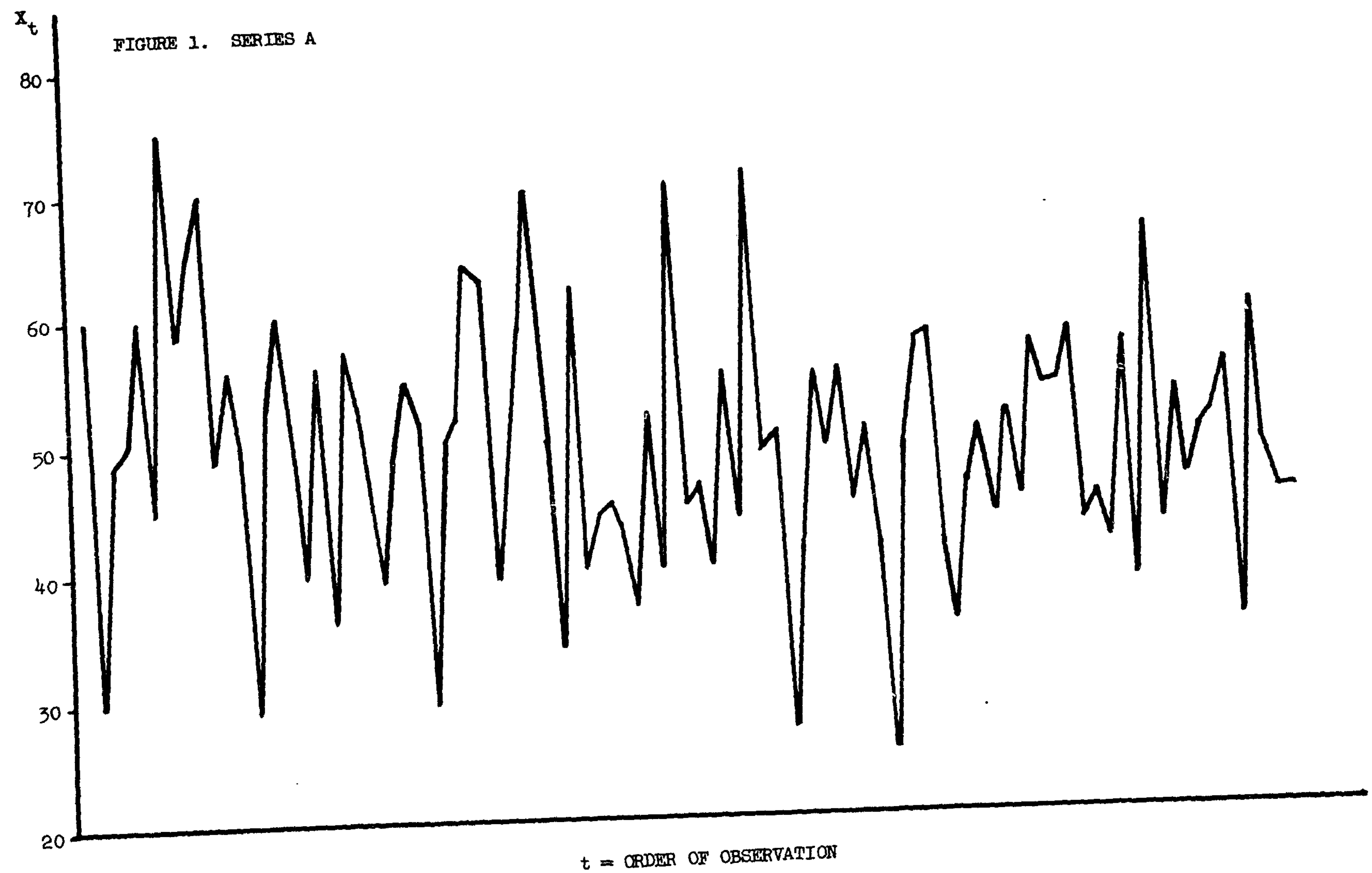
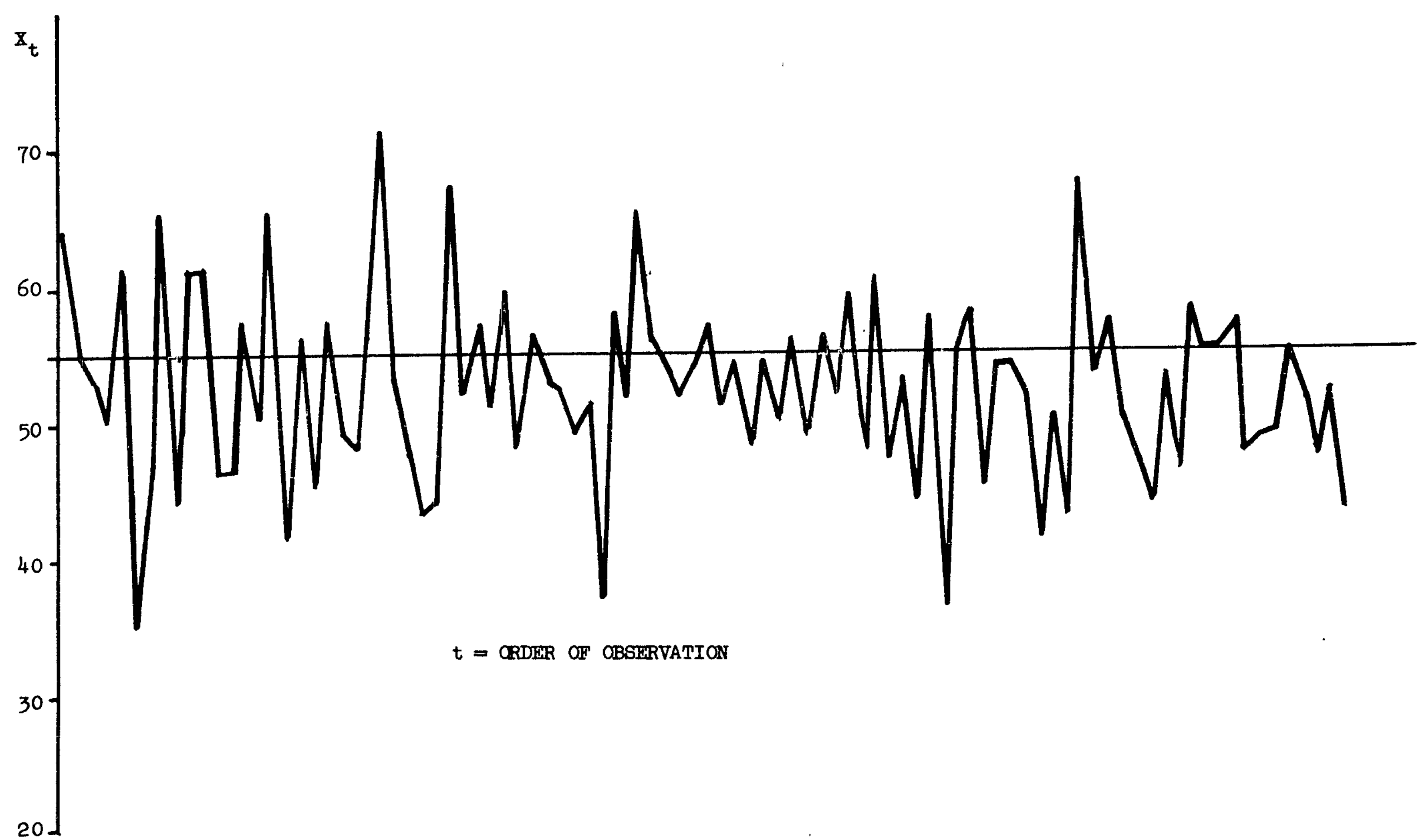


FIGURE 1. SERIES B



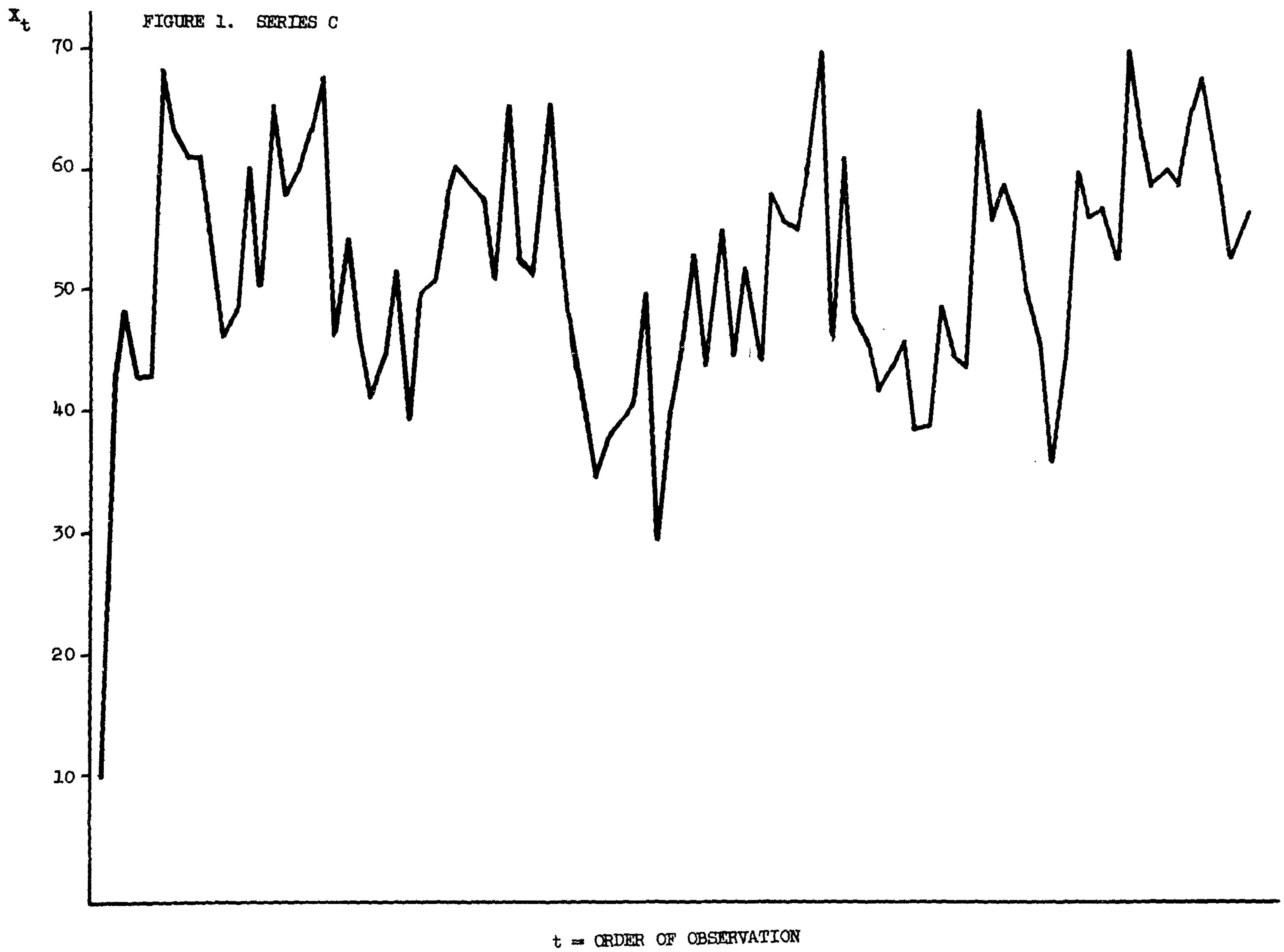
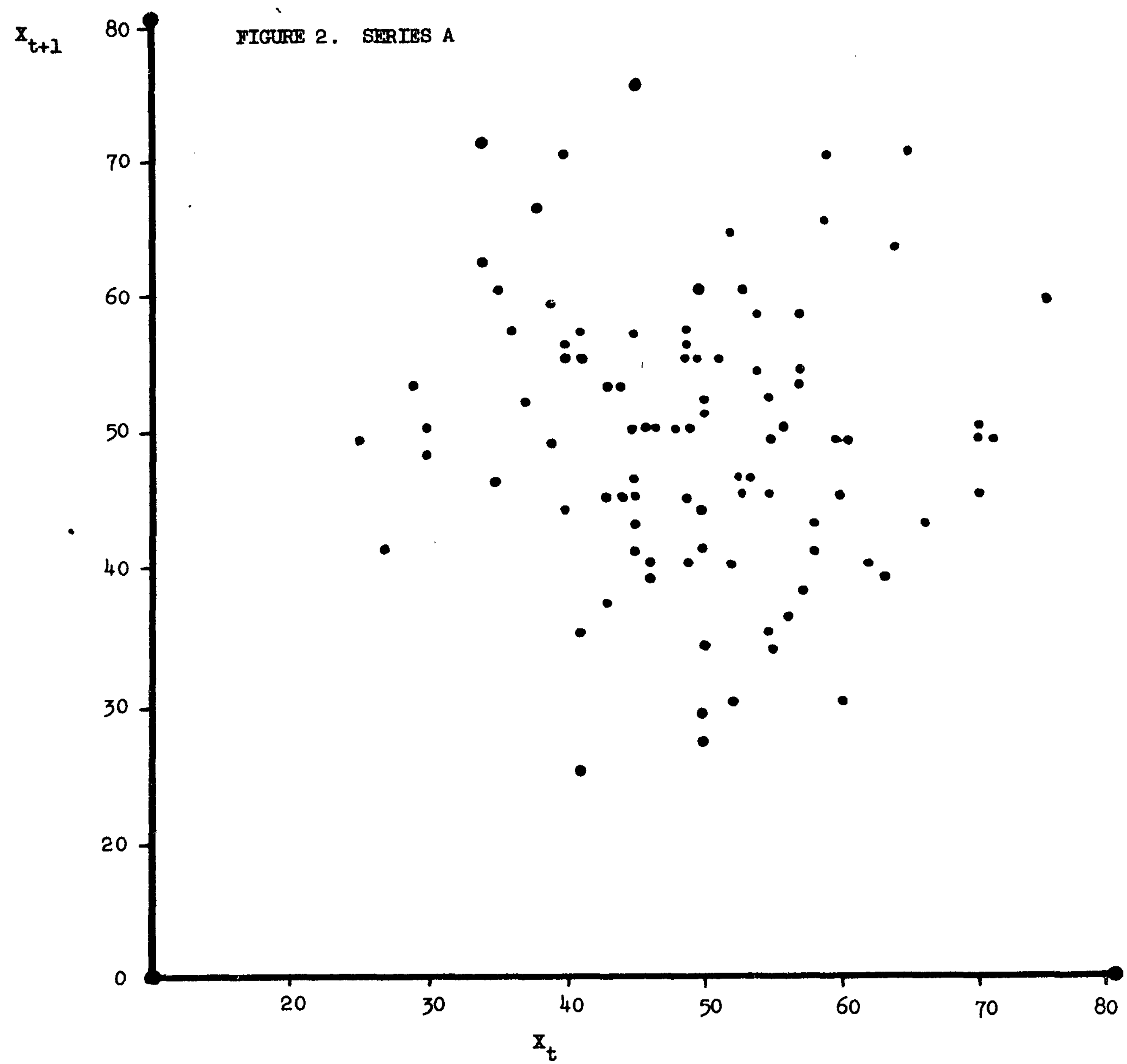


FIGURE 1. SERIES C

FIGURE 2. SERIES A



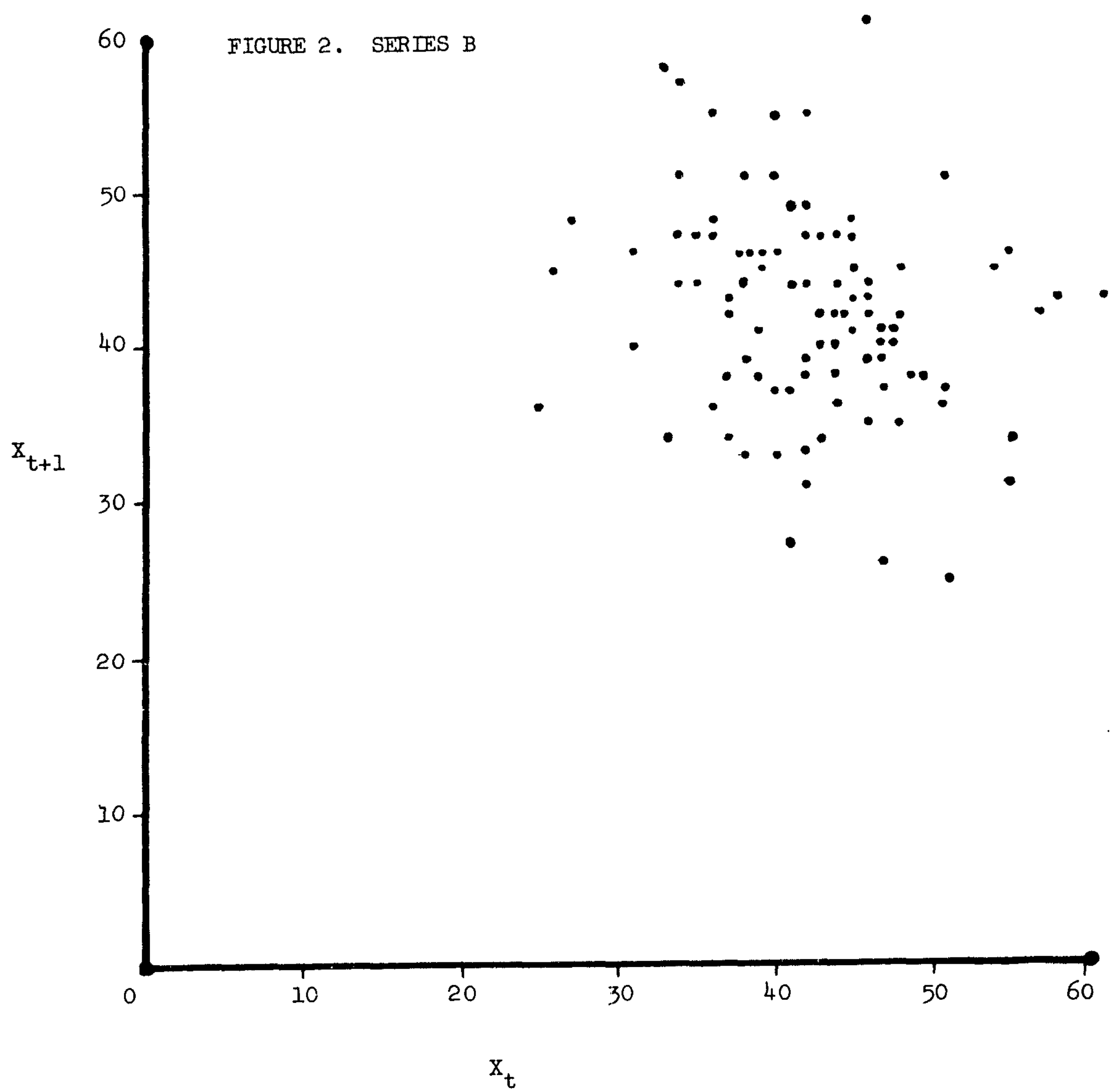


FIGURE 2. SERIES B

FIGURE 2. SERIES C

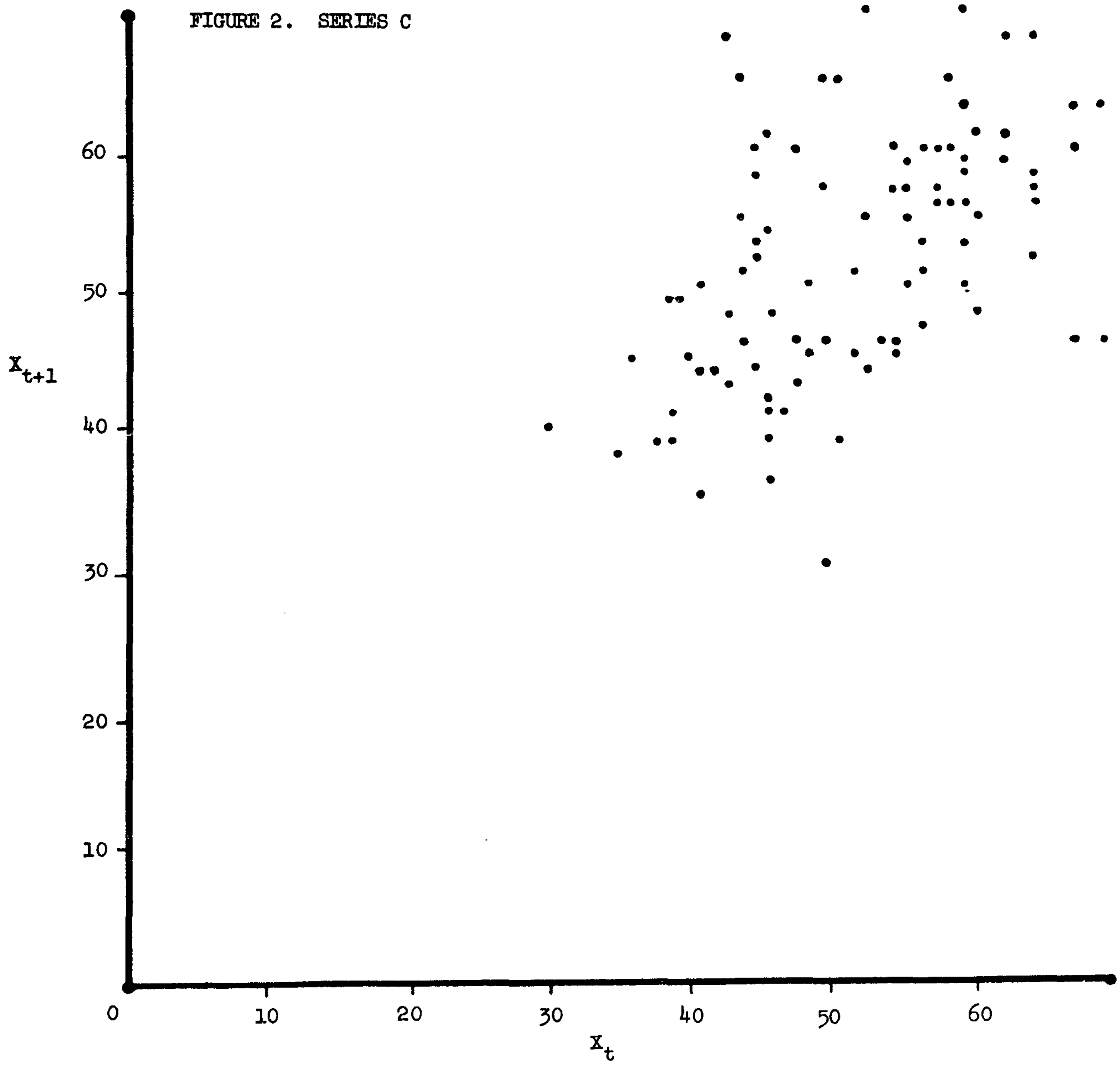
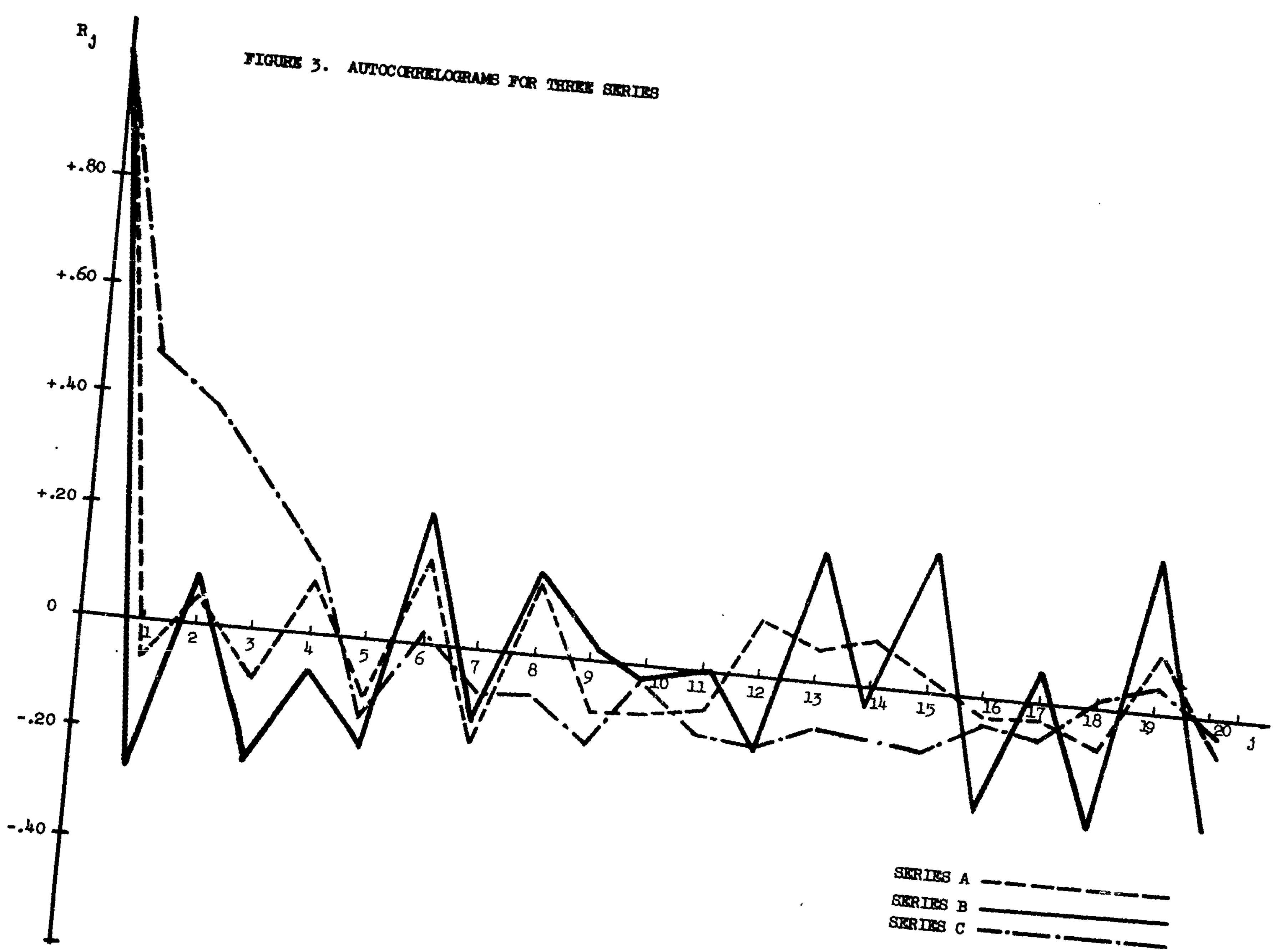


FIGURE 3. AUTOCORRELOGRAMS FOR THREE SERIES



composed of two lags of one observation each. The general constraint on the R's is given by the relationship:

$$(3.4) \qquad \begin{vmatrix} 1 & R_1 & R_2 & R_3 & \cdots\cdots & R_s \\ R_1 & 1 & R_1 & R_2 & \cdots\cdots & R_{s-1} \\ R_2 & R_1 & 1 & R_1 & \cdots\cdots & R_{s-2} \\ R_3 & R_2 & R_1 & 1 & \cdots\cdots & R_{s-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ R_s & R_{s-1} & R_{s-2} & R_{s-3} & \cdots\cdots & 1 \end{vmatrix} \geq 0, \text{ for any value of } s$$

In view of these constraints, the information provided by the R's for the longer lags is subject to diminishing return; that is, it is not necessary to know <u>all</u> the R's. A limited number of them -- R_1, R_2, R_3, ...R_s -- where s is small compared to N, will generally suffice to give a nearly-complete summarization of all the linear dependencies in the data and thus a nearly-complete summarization of the time-ordered process itself. That is the power of this form of summarization. Unfortunately, sampling fluctuations will always mask the true values of R. The sampling properties of R_j have been studied by Anderson (1) and others (11) (41). Assuming N-variate normality of the parent population $\{X_t\}_{t=1,2....N}$, an R_j whose true value is ρ_j will be asymptotically normally distributed with mean ρ_j and standard deviation $\frac{1}{\sqrt{N-j}}$. This much is known, but the complicated constraints upon the R_j (see (3.4)) make a joint significance test upon two, ten or twenty of them an unmanageable affair. Over and above the consideration of the difficulty of such a test, the R_j have the unhappy property (due to the constraints again) that a sampling peculiarity in one of the first few lags will intrude itself artificially into the later lags. Thus the over-all picture given by the R's is made blurry, as Kendall himself has complained (23).

IV

OTHER METHODS: SPECTRAL ANALYSIS

What is desired is a descriptive scheme with the advantage of nearly-complete summarization provided by autocorrelational analysis, yet without the disadvantageous sampling properties of that scheme. Spectral analysis (39) is just such a scheme. Related analytical techniques, such as Fourier analysis, harmonic analysis, etc. have been used in one form or another in many scientific fields where time-ordered data occur, particularly in electrical engineering (38).

The essential idea of spectral analysis is that any stationary time-series can be decomposed into a large number of cosine-wave components. That is, the oscillations in the time-series are accounted for by a sum of cosinusoidal oscillations of varying frequencies. All frequencies are admissible, and the number of component waves is taken to be infinite; this represents an advance over the naive notion that a small number of components with frequencies commensurate with the unit of observation on the time scale are adequate to account for most empirical data. The amplitudes of the component waves are an important consideration. The bigger the amplitude of a component wave, the more important it is in the explanation of the time-series. In fact, the <u>amplitudes of the component waves</u> are related to the <u>total variance of the time-series</u> in a very neat and useful way, as shall be shown below. The <u>phases</u> of the cosine components are essentially irrelevant to spectral analysis, except insofar as they allow for consideration of <u>statistical variation</u>. It will be remembered that a distinction was drawn between a time-series sample and a time-series population. Different samples from the same population differ from each other;

spectral analysis assumes that the differences between samples are due to the fact that the various cosine components in the several samples have different phases.

These various ideas can be stated mathematically as follows:

$$\{X_t\} = \sum_{i=1}^{\infty} a_i \cos(w_i t + \phi_i) \tag{4.1}$$

where $\{X_t\}$ is the population of X's at any single value of t,

a_i is the (fixed) amplitude of the i:th component wave,

w_i is the (fixed) frequency of the i:th component wave,

ϕ_i is the (variable) phase of the i:th component wave,

and the summation indicates that an infinite number of such components are being added together.

It is assumed that the phases ϕ_i vary statistically from sample to sample; the ϕ_i for different components, i, are assumed to vary independently of one another, and the distribution of each ϕ_i is assumed to be uniform (rectangular) over the interval 0 to 2π (i.e., all values of ϕ_i from 0 to 2π have equal likelihood of occurrence in any given sample).

The form (4.1) is a perfectly general representation of any stationary[4] time series. Any function with values defined at the points t=1 through t=N can be fitted exactly at those points by a unique series of

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[4]The fact that (4.1) represents a stationary time-series can be seen by substituting t+k for t in that formula. Then $\{X_{t+k}\} = \Sigma\, a_i \cos(w_i [t+k] + \phi_i) = \Sigma\, a_i \cos(w_i t + [\phi_i + w_i k])$. Define $\theta_i = \phi_i + w_i k$. Now θ_i has the same distributional properties as ϕ_i, since w_i and k are fixed. Thus time translation does not affect the population.

the form (4.1). This is a classical result of Fourier analysis. Under certain conditions, the series will not agree with the function at one or more points which we do not observe. But as far as what we observe is concerned, any stationary time-series acts as though the underlying population were a sum of an infinite number of cosine components. That is the power of this formulation.

The total variance of the time-series population (i.e., the average squared deviation of all possible X's from the expected value of X at any fixed value of time) is given by

$$(4.2) \qquad \sigma_X^2 = \sum_{i=1}^{\infty} \frac{1}{2} a_i^2 \quad .$$

(The derivation of this result is given in Appendix A.) That is, each component wave of frequency w_i contributes to the total variance in proportion to the square of its amplitude, a_i. The contribution to the total variance of all waves with frequencies between w_1 and w_2 is equal[5] to $\sum_{w_1 < w_i < w_2} \frac{1}{2} a_i^2$. The contribution to the total variance of all waves with frequency less than some w has a special significance. Denote this quantity by S(w), the so-called integrated spectrum.

$$(4.3) \qquad S(w) = \sum_{w_i < w} \frac{1}{2} a_i^2 \quad .$$

The extreme values of this function are $S(0) = 0$ and $S(\infty) = \sigma_X^2$. The

[5]Spectral analysis does computationally with a discrete series of observations subject to sampling fluctuations what a wave filter does to an electrical current. In this analogy, the total variance of a time-series is equivalent to the total power in an electrical discharge.

function increases monotonically as w goes from 0 to ∞ . Indeed, with an infinite number of admissible component frequencies, contributions to the total variance may be considered to occur at all frequency values, and S(w) may be taken to be a continuous function. In that case, S(w) has a derivative at all values of w. Call this derivative s(w). s(w) is called the <u>spectrum</u>. To every stationary time-series population there corresponds a unique s(w), with the property that the contribution to the total variance by waves of frequencies between w_1 and w_2 is equal to $\int_{w_1}^{w_2} s(w)\,dw$.

Thus far no mention has been made of the fact that we are interested in dealing with a finite, discrete set of observations. This condition imposes an upper limit on the range of admissible frequencies, w, because very high frequency components are not detectable by observation. They undergo their fast wiggles in between the points of observation, and as far as what we observe is concerned, a high frequency component will appear as some lower, observable frequency. This is the phenomenon of the <u>alias</u>, pictured in Figure 4A. Figure 4B gives the exact relationship between high-frequency components and their low-frequency "aliases." The highest observable frequency is one cycle per every two observation points. This corresponds to the angular frequency $W = \pi$. Thus the limits on the admissible range of frequencies are $0 \leq w \leq \pi$, and the spectrum s(w) is confined to that range.

In order to estimate s(w) from a single time-series sample, it is necessary to compute estimates of s(w) at several values of w, and then sketch in the rest of the curve. These estimates are ordinarily computed at equally spaced values of w; if the total number of estimates is (m+1),

FIGURE 4A. HOW THE 'ALIAS' WORKS

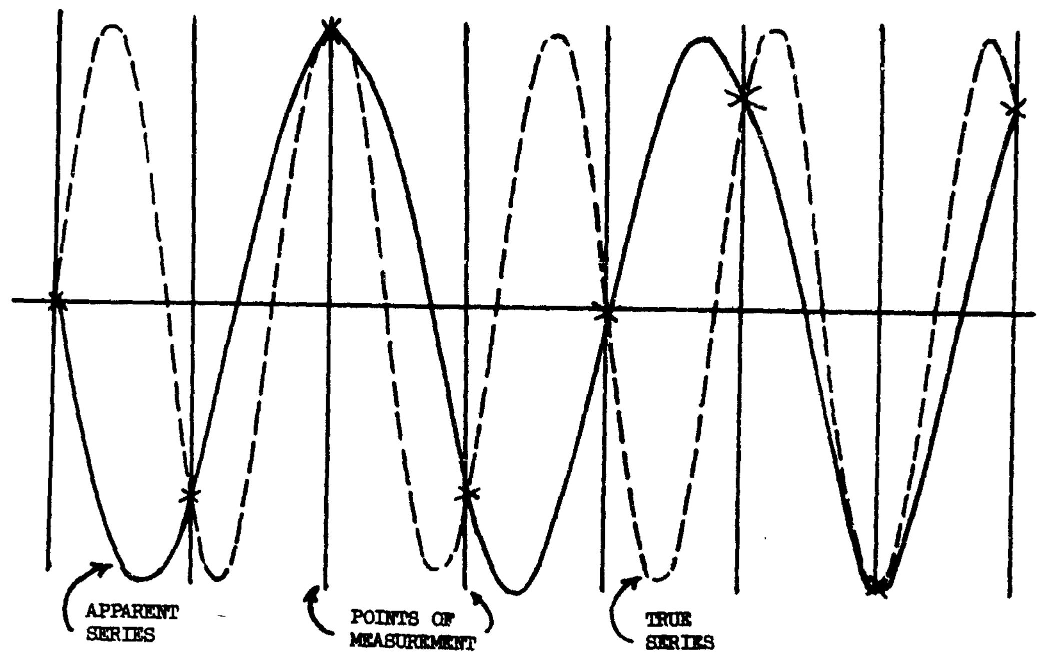
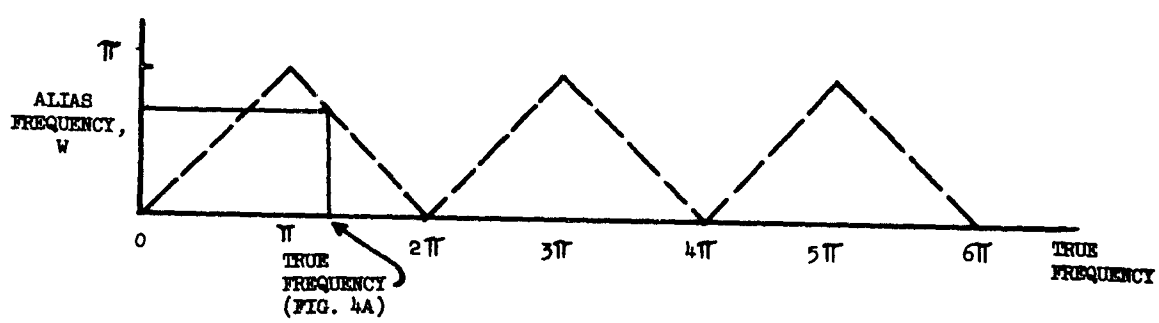


FIGURE 4B. ALIAS RELATIONSHIPS



the individual estimates are made at $w = 0, \frac{\pi}{m}, \frac{2\pi}{m}, \frac{3\pi}{m}, \ldots\ldots \frac{(m-1)\pi}{m}, \pi$. These individual estimates are denoted by U_p, $(p = 0,1,2\ldots m)$ the estimated spectral densities at frequencies defined by $w = \frac{p\pi}{m}$. The $(m+1)$ values of U_p provide as near-complete a summarization of the time-series data as do the first $(m+1)$ lag covariances Q_j. In fact, the U_p are computed from certain linear combinations of the lag covariances, Q_j $(j=0,1,2,..m)$, a different combination for each value of p.

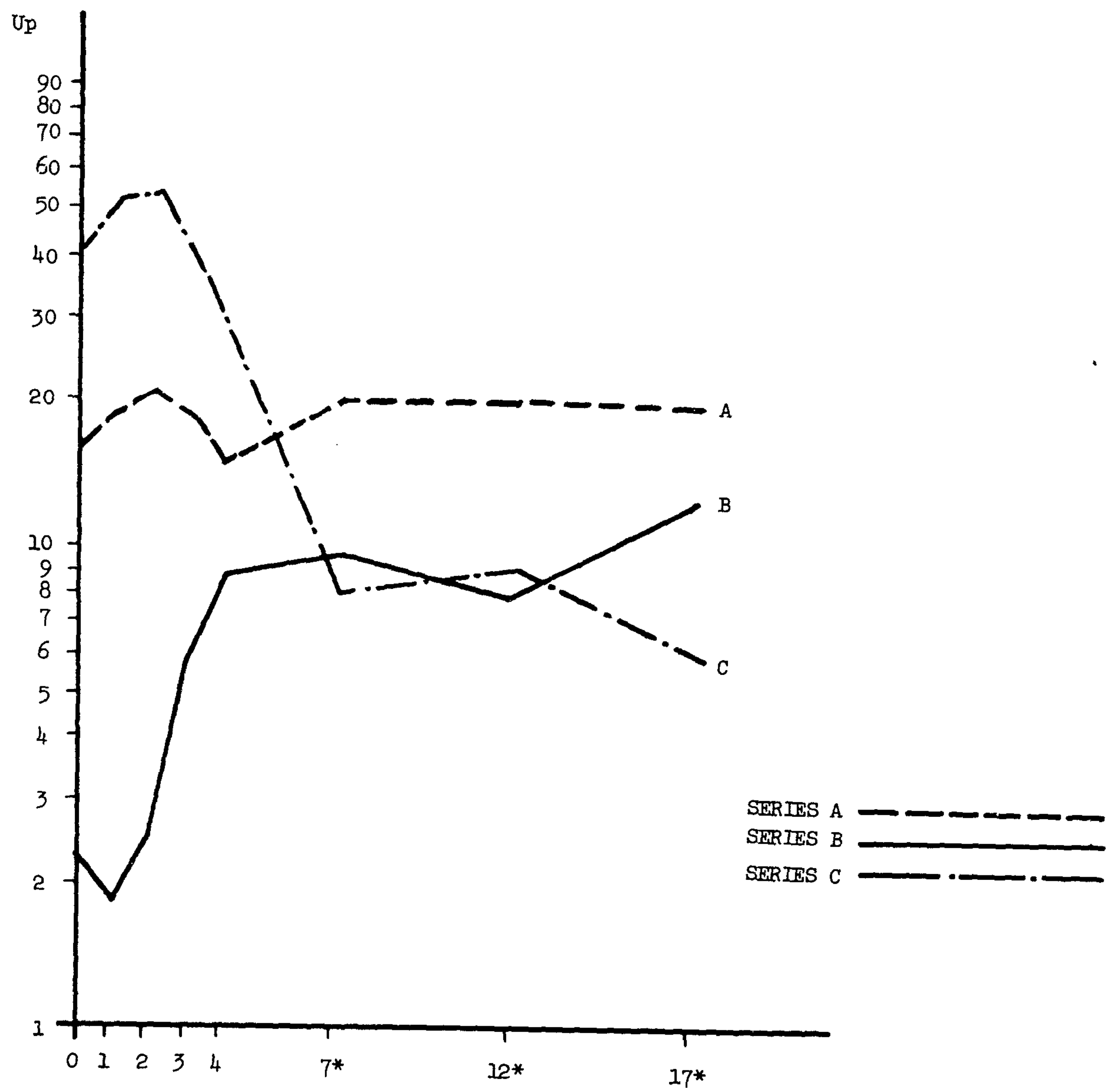
$$(4.4) \qquad U_p = \sum_{j=0}^{m} T_{jp}Q_j \quad .$$

The T_{jp} are presented in Appendix B along with the rationale for this computational device.

For a random series, the spectrum $s(w)$ is flat over the range 0 to π (i.e., $s(w)$ = constant).

Deviations of the spectrum from a constant level indicate departures from randomness. If $s(w)$ increases as w goes from 0 to π, the time-series will tend toward sharp quick ups and downs (high frequency components). If $s(w)$ decreases as w goes from 0 to π, the time-series contains more low frequency components than high frequency components; that is, it tends toward slow, wandering trends and cycles. (See Figure 5.) Another type of possibility is that the spectrum will consist only of a very sharp peak at $w=w_o$. (In other words, the time-series does not decompose into many components--there is only cone component, at $w=w_o$.) This situation is very nearly true in certain gross physical phenomena, such as tides and sunspots. But it is less often true than one might expect, and the search for sharply defined frequencies in any and all time-series has

FIGURE 5. SPECTRA FOR THE THREE SERIES OF FIGURE 1



been largely a failure. There is a method called periodogram analysis which sets out to find such well-defined frequencies (see Davis (9) for an excellent historical summary and Wold (43) for a mathematical treatment). Periodogram analysis can be treated as a restricted case of spectral analysis. Controversies about the exact periodicities presumed inherent in business phenomena have not been settled by periodogram analysis, because periodicities (or frequencies; the two are reciprocal to each other) are seldom sharp and exact. They are blurred over some range. For example, the periodicity of the short business cycle is said to be between 36 and 48 months (9). Certainly, in complex psychological data, sharply defined frequencies are not to be expected. There are usually too great a number of factors acting for a single well-defined cosine component to dominate a phenomenon. The major difficulty with studies of psychological rhythms is that in talking of a rhythm, we fail to appreciate the fact that the so-called rhythm may be so blurred (the peak on the spectrum may be so broad) that the phenomenon is nearer to randomicity than to rhythmicality. (See Section I and Bills' study, Section V.)

The important difference in utility between spectral analysis and autocorrelational analysis is that the statistical properties of the estimates of spectral density are very convenient; sampling variations can be handled much more easily by spectral analysis. These statistical properties have been worked out recently by Professor John Tukey (39) and are stated below:

(4.5) If the true (population) value of U_p is $\hat{U}_p$, the quantity $\left[f \frac{U_p}{\hat{U}_p} \right]$

is approximately distributed as χ^2 with f degrees of freedom, where $f = \frac{2N}{m} = \frac{2\ (\text{number of observations in the time series})}{(\text{number of lag covariances computed})}$. (There is a slight loss of degrees of freedom when the spectrum departs considerably from flatness. See footnote 9, page 56.)

A convenient statement for graphical purposes is that $\log U_p - \log \hat{U}_p$ is distributed as $\log \frac{\chi^2}{f}$. In other words, on a simultaneous plot of $\log U_p$ vs. p and $\log \hat{U}_p$ vs. p, the distance between the points at any value of p is a direct measure of the probability of occurrence of the sample value U_p under the hypothesis that $\hat{U}_p$ is the true value. The standard procedure used throughout this paper is to plot spectral densities on semi-logarithmic paper, with U_p as the (logarithmic) ordinate and p as the abscissa. Any set of empirical values of U_p can then be surrounded by a confidence region, corresponding to some probability level, which delimits the area within which the population values, $\hat{U}_p$, will probably lie.

(4.6) If two samples (one of them denoted by ') from the same time-series population are drawn, the quantity $\frac{U_p}{U'_p}$ is approximately distributed as F with $\frac{2N}{m}$ and $\frac{2N'}{m'}$ degrees of freedom. For this significance criterion, the semi-logarithmic plot is also convenient.

Property (4.5) is additive over several samples. In other words, $\left[f_1 \frac{U_{p1}}{\hat{U}_{p1}} + f_2 \frac{U_{p2}}{\hat{U}_{p2}} + \ldots + f_r \frac{U_{pr}}{\hat{U}_{pr}}\right]$ is approximately distributed as χ^2 with $f_1 + f_2 + \ldots + f_r$ degrees of freedom. Property (4.6) is also additive, but the addition must be done separately for numerator and denominator:

$\frac{U_{p1} + U_{p2} + \cdots\cdots\cdots + U_{pr}}{U'_{p1} + U'_{p2} + \cdots\cdots\cdots + U'_{pr}}$ is approximately distributed as F with

$\frac{2N_1}{m_1} + \frac{2N_2}{m_2} + \ldots\ldots + \frac{2N_r}{m_r}$ and $\frac{2N'_1}{m'_1} + \frac{2N'_2}{m'_2} + \ldots\ldots + \frac{2N'_r}{m'_r}$ degrees of freedom.

(4.7) Adjacent values of U, U_p and U_{p+1}, are not independent, due to computational blurring. However, any U_p and U_{p+2} are independent. A very high value of U will spill over somewhat into the neighboring values, but no further. The over-all effect is a slight smoothing of the spectrum.

The advantage of spectral analysis over autocorrelational analysis is the relative lack of constraint between U values and the simplicity and utility of their sampling properties. Furthermore, a spectrum presents a much clearer graphical picture than an autocorrelational plot. To cite a specific analytical example on the basis of which we may compare spectral and autocorrelational analysis, consider a simple case of the so-called auto-regressive scheme (see Yule and Kendall (44)). By this scheme, each observation, X_t, is determined by a linear combination of the previous observation and an independent random error. This may be written

(4.8) $X_t = aX_{t-1} + \mathcal{E}_t$, where a is a constant between -1 and +1. It can be shown that the autocorrelations, R_j, for such a scheme are given by

(4.9) $R_j = a^j$.

The spectrum, meanwhile, is

$$(4.10) \qquad s(w) = \frac{1}{(1+a^2) - 2a\cos w} \quad .$$

An empirical data series generated by such a scheme with $a = .5$ is shown in Figure 6A. Figure 6B gives the theoretical and empirical autocorrelation plots for this series. Figure 6C gives the theoretical and empirical spectral plots for the same series. The picture the spectrum presents is much clearer than the autocorrelational picture.

In summary, spectral analysis decomposes the total oscillation of a time-series into cosine oscillations of varying rates, specifying the relative contribution of each of the components of oscillation. It does this by a unique function $s(w)$ which conveniently summarizes these relative contributions as a function of the frequency of oscillation w, of the contributing components. The statistical sampling properties of the estimates, U, of $s(w)$ are known, enabling us to assess whether a time-series sample has come from a particular time-series population or to determine whether two time-series samples have come from the same time-series population. Spectral analysis is thus superior to autocorrelational analysis as an analytical method, and will be used to deal with the experimental data of this paper.

FIGURE 6A. (AN AUTOREGRESSIVE SERIES)

$X_t = \frac{1}{2}X_{t-1} + \varepsilon_t$

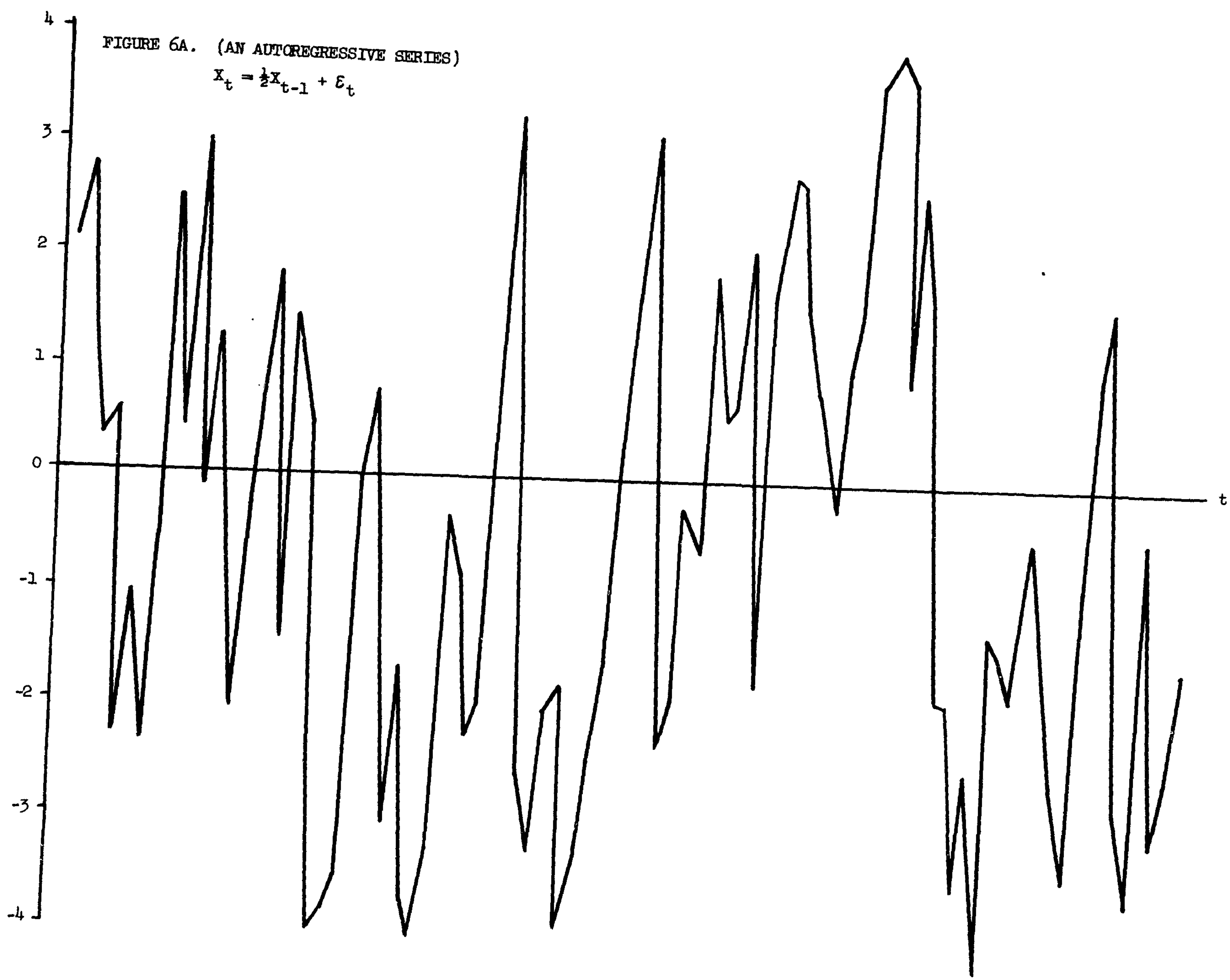


FIGURE 6B. AUTOCORRELATIONS FOR THE SERIES OF FIGURE 6A

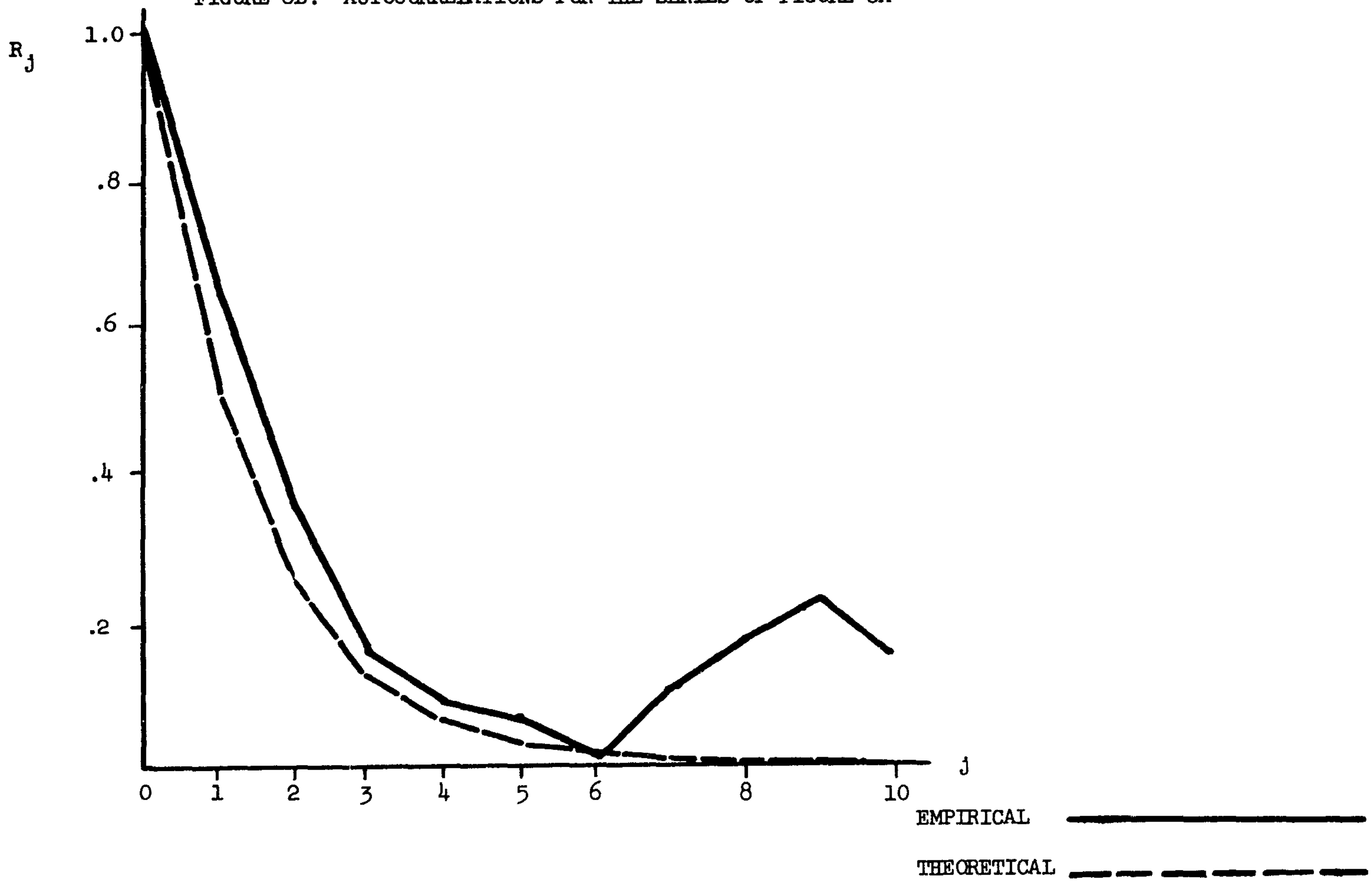
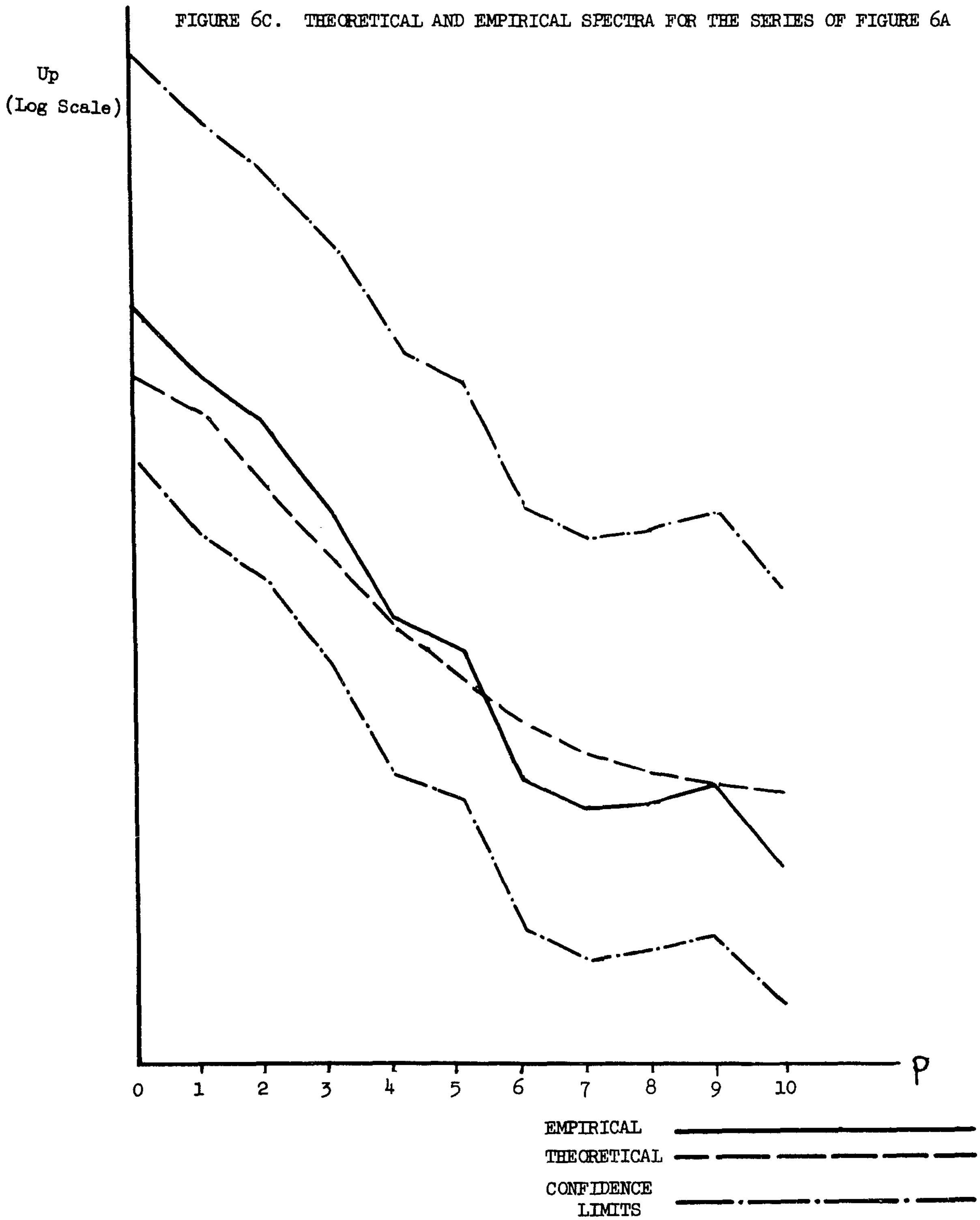


FIGURE 6C. THEORETICAL AND EMPIRICAL SPECTRA FOR THE SERIES OF FIGURE 6A



V

SPECTRAL ANALYSIS APPLIED TO PREVIOUS PSYCHOLOGICAL DATA

In order to confirm the suitability of spectral analysis for analyzing time-ordered psychological data, the method will be applied to the data of two previous experimental studies. In so doing, we shall incidentally clarify certain of the conclusions made by the authors in those studies. The two chosen studies are those of Bills (3) on mental blocking, and Day (10) on serial patterns of responses in the auditory threshold.

A. A Study of "Mental Blocking"

Bills (3) gave his subjects simple, repetitive tasks to perform, essentially involving an enforced rapid choice amongst alternatives. For example, a long row of randomly intermingled yellow, blue, red and green dots was shown with the instruction to read from left to right, calling off the colors of the dots as rapidly as possible. The response times of each successive response were recorded in serial order. Bills arbitrarily considered a response to be "blocked" when the response time was more than twice the length of the modal reaction time prevailing during the minute of work when the block occurred. Two questions are then asked of the data:

1. Are the blocks real; i.e., qualitatively different from normal responses, and not simple extreme variations within a unimodal distribution of response times.

2. Are the blocks "rhythmic" or periodic in occurrence.

Bills answers the first question in the affirmative by presenting a distribution of response times which looks bimodal. However, the number of cases is much too small: only 4 cases at the secondary mode as compared to 2 cases at the presumed dip between the modes. Apparent bimodality is a notorious snare and delusion, since it can so readily appear by chance with small samples from a unimodal distribution. Good statistical evidence for bimodality is not presented by Bills.

Bills approaches the problem of the rhythmicality of the blocks by plotting a frequency distribution of the distances (in number of responses) between adjacent blocks. The following statement is made concerning this frequency distribution: "If no periodicity exists, the frequency curve of interblock distances should show a normal or chance dispersion. If, on the other hand, a definite periodicity exists, it should emerge as a mode at some definite point along the base line, with a narrow range of dispersion of the measures from it. If more than one periodicity exists, there should be a multimodal distribution of the interblock distances, with a separate mode for each periodicity." (6). The first part of this statement is in error, and the last part is highly questionable. If no periodicity exists, the distribution of response times is not normal, but decreases geometrically. Where p is the probability of occurrence of a block on any single response, the probability of an interblock interval of N responses is $\left[p(1 - p)^{N-1}\right]$. With this erroneous statement in hand, Bills presents several experimental frequency distributions of interblock responses and claims that two periodicities are revealed by each of these records (4). There is no sampling theory given, nor are there tests of statistical significance. The conclusions are based solely on the apparent modes in the frequency distribution, an inadequate method of analysis.

Spectral analysis was applied to a set of 75 successive response times on a figure analogies task. The data, shown in Figure 7, were originally presented in Bills' chapter in Andrews (6) as typifying the blocking phenomenon. Spectral analysis cannot answer (directly) the question of the bimodality of the distribution of response times, since this is based on discrete and not serial information. The question of the rhythmicality of blocks can, however, be answered directly and clearly by spectral analysis.

The estimates of spectral density of the blocking data are presented graphically in Figure 8 on a semi-logarithmic plot (see p. 23). These estimates U_p, were computed on the basis of the first 20 lag covariances, Q_1 through Q_{20}, and the variance Q_o. Estimates were computed only at frequencies $w = 0, \frac{\pi}{20}, \frac{2\pi}{20}, \frac{3\pi}{20}, \frac{4\pi}{20}, \frac{7\pi}{20}, \frac{12\pi}{20}, \frac{17\pi}{20}$, (i.e., $p = 0,1,2,3,4,7,12$, and 17), due to a short-cut computational method explained in Appendix C. The estimates at $p = 7,12$, and 17 actually each represent averages of five adjacent U_p. Ninety per cent confidence limits, based on the χ^2-test given in (4.5) are placed above and below the empirical curve. This confidence belt can be expected to include the true, or population spectrum at 90% of the values of p (frequency) for which U's have been computed. It will be noted that a flat spectrum fits very nicely into this confidence belt. In other words, the series of 75 response times is not significantly different from a random series; the evidence for rhythmicality is nil. The apparent dip and the later drop-off in the spectrum of the empirical data cannot be considered as other than phantoms of sampling. On the other hand, if blocks had occurred in these data with perfect rhythmic regularity and if the series were in all other

FIGURE 7. BILLS' BLOCKING DATA

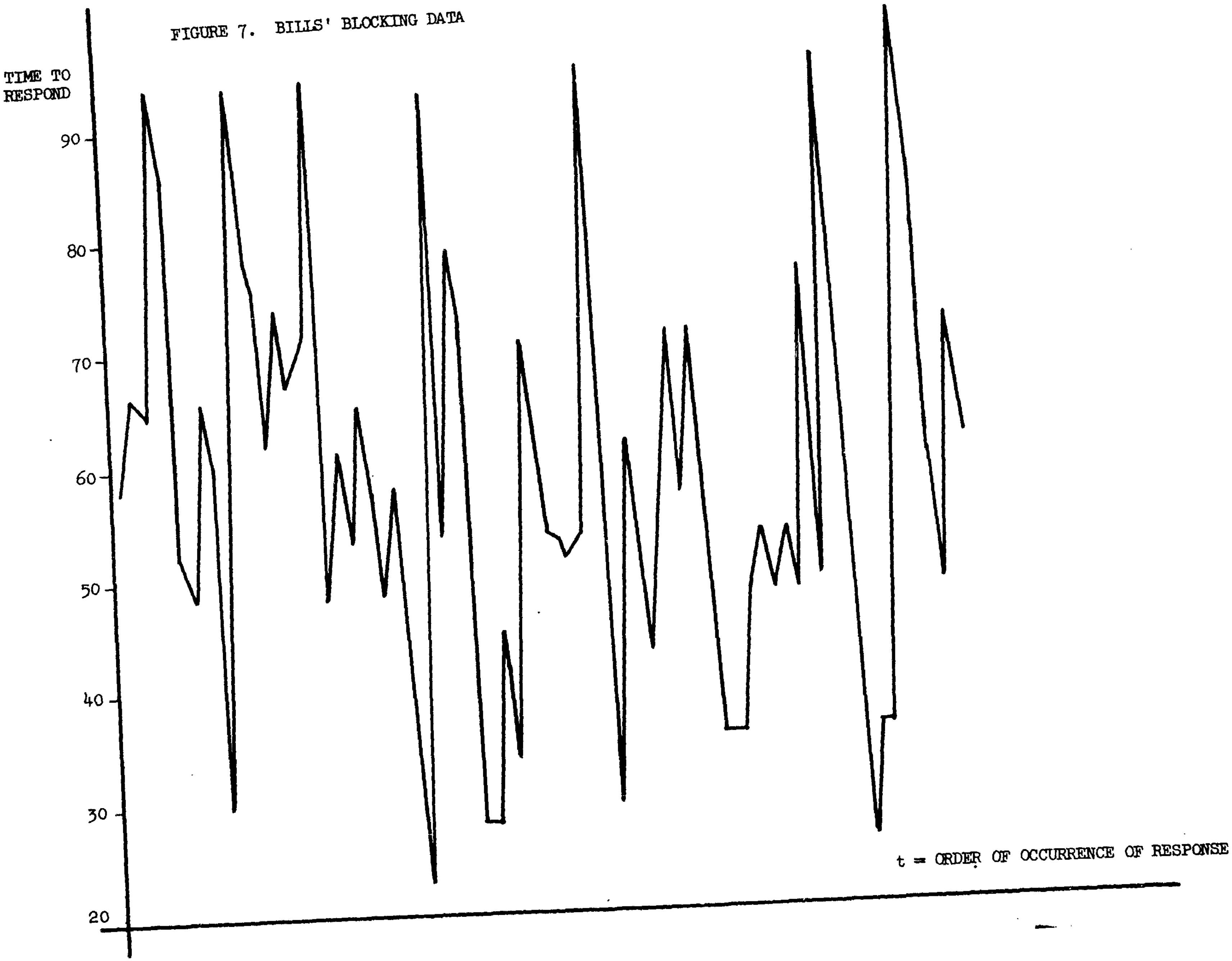
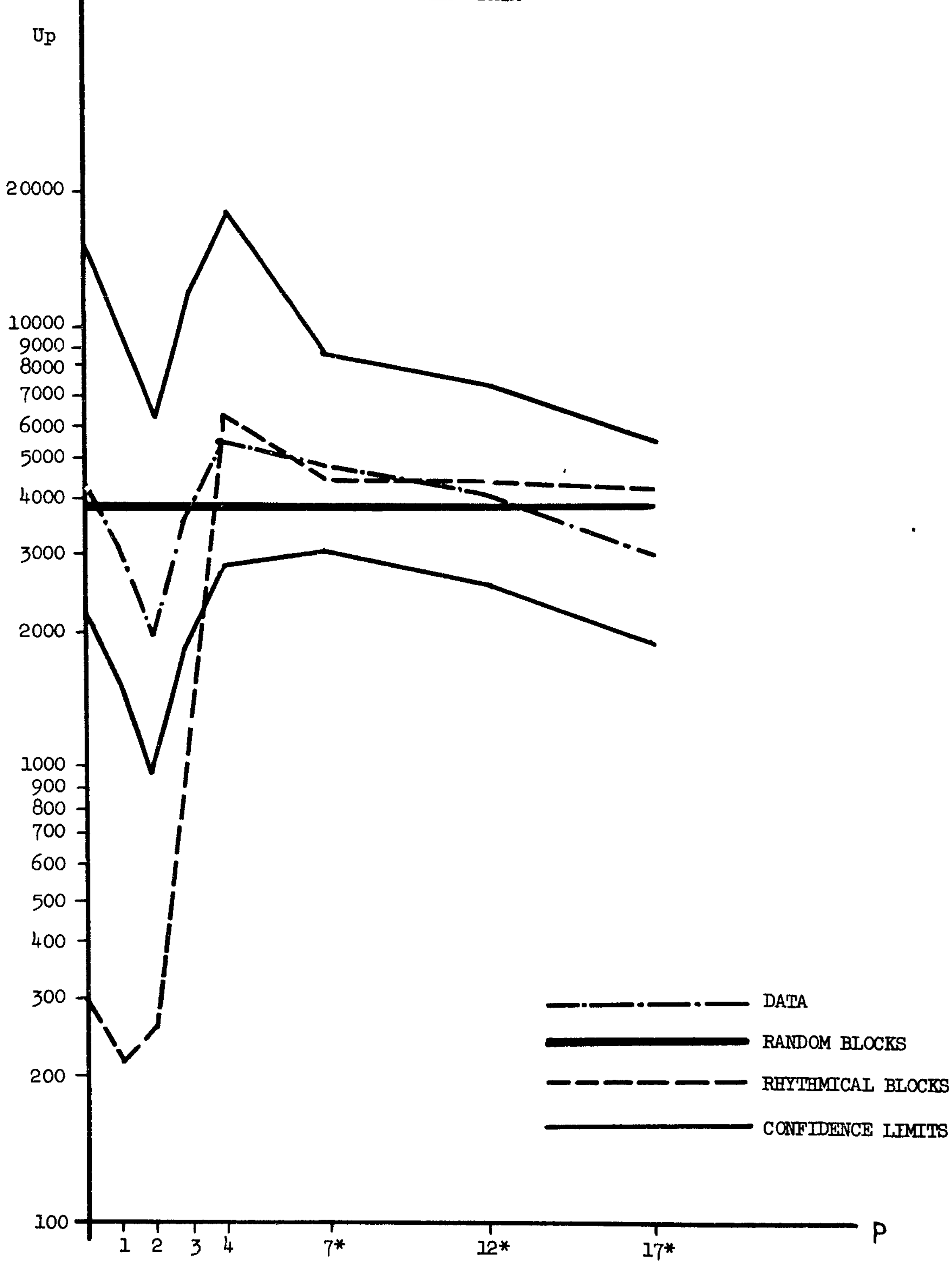


FIGURE 8. SPECTRA FOR BILLS' DATA



respects random, the theoretical spectrum would have been as indicated by the dotted line. Conformity of the data to this spectrum is close at high frequencies, but at p = 0,1,2,3 the discrepancy is enormous. In all fairness to Bills, it is true that this analysis pertains to only one fragmentary portion of the data he has collected. But analysis of this nature is necessary before statements about rhythmicality can be sensibly made, and, indeed, such analysis should be carried out on more extensive data.

B. <u>On Serial Patterns in Auditory Discrimination Judgments</u>

We have seen spectral analysis applied to what was essentially a random series. The use of spectral analysis considerably altered the conclusions which had been drawn by the original investigator, who used inspection as his only technique. Let us now turn to the case of a clearly non-random time series upon which a refined objective technique (autocorrelational analysis) had already been brought to bear by the previous investigator. Does spectral analysis add anything to the detail and precision of the previous analysis?

Day (10), acting on a lead supplied by Flynn (17), has analyzed the tendency for subjects to respond in non-random sequential patterns in auditory discriminations.[6] A 1000 cycle, 16 db. tone was sounded steadily except for a brief temporary increase in intensity occurring every T seconds. The duration of these pips was .1 second and their amount was fixed for each individual subject at his particular differential threshold (50% level) for the given frequency and intensity.

---

[6] This particular line of experimentation is very old, and joins historically with research on the attention wave. But it is only recently that adequate mathematical treatment has been employed.

The subject pressed a key whenever he perceived an increase in intensity. The data consisted of a series of "yesses" and "nos," a "yes" whenever the subject heard the actual increase, a "no" whenever he did not. No mention is made in Day's paper of instances in which the subject "heard" an increase when it wasn't there; presumably this was a rare occurrence. The value of T, the interval between successive pips, was held constant at each sitting, but varied from sitting to sitting. Five values of T were used for each of the five experimental subjects. There were: 1.6 seconds, 2.1 seconds, 4.2 seconds, 7.1 seconds, and 10.6 seconds. The numbers of observations per subject recorded for each interval were 600, 600, 450, 350 and 300 respectively. Each series of yesses and nos was analyzed by an essentially autocorrelational analysis. Each "yes" was designated by a "1," each "no" by a "0," and the first 20 lag correlations were computed from an index devised by Flynn. Flynn's index for the j:th lag is defined as

$$(5.1) \quad \Omega_j = \frac{\text{Observed no. of "matches" - Expected no. of "matches"}}{\text{Maximum no. of "matches" - Expected no. of "matches"}},$$

where a "match" at the j:th lag occurs whenever two 1's occupy positions j responses apart or two 0's occupy positions j responses apart.

This index can be shown to be approximately equal to the ordinary product-moment lag correlation under certain conditions, as follows:

Let N = the number of observations in the series

N_1 = the number of 1's in the series; N_0 = the number of 0's in the series

A_j = the number of matching pairs of 1's at lag j in the series

B_j = the number of matching pairs of 0's at lag j in the series

Assume 1. That N is very large.

2. That the number of 1's in the series is equal to the number of 0's, as it must to satisfy the specification that the stimulus is just at the threshold. $N_1 = N_0 = \frac{N}{2}$. From this it follows indirectly that $A_j = B_j$.

Now the observed number of matches is $(A_j + B_j) = 2A_j$; the maximum number of matches is $(N - j)$, the total number of pairs at the j:th lag; and the expected number of matches is $\frac{1}{2}(N - j)$, since in a random series a match occurs with probability $\frac{1}{2}$. Finally, neglecting j with respect to N, the formula for Ω_j can be simplified to

$$(5.2) \qquad \Omega_j = \frac{2A_j - \frac{N}{2}}{N - \frac{N}{2}} = 4\frac{A_j}{N} - 1 \quad .$$

But the product-moment lag correlation, R_j, is computed from $R_j = \frac{Q_j}{Q_0}$, where from (3.2),

$$Q_j = \frac{\sum_{t=1}^{N-j} X_t X_{t+j}}{N - j} - \frac{\left[\sum_{t=1}^{N-j} X_t\right]\left[\sum_{t=1}^{N-j} X_{t+j}\right]}{(N - j)^2} \quad .$$

Now, since the X's must be either 0 or 1, there are $\frac{N}{2}$ of each, and N is large enough so that j can be neglected in comparison to N, the formula for Q_j simplifies to

$$(5.3) \qquad Q_j = \frac{A_j}{N} - \frac{1}{4} \quad .$$

$$(5.4) \quad \text{But} \qquad Q_0 = \frac{A_0}{N} - \frac{1}{4} = \frac{\frac{1}{2}N}{N} - \frac{1}{4} = \frac{1}{4} \quad .$$

Therefore

$$(5.5) \qquad R_j = \frac{Q_j}{Q_0} = 4\left(\frac{A_j}{N} - \frac{1}{4}\right) = \frac{4A_j}{N} - 1 = \Omega_j \text{ , by (5.2).}$$

Day's sets of autocorrelations (plotted in Figure 9) demonstrate the following properties:

1. For T = 1.6 and T = 2.1,

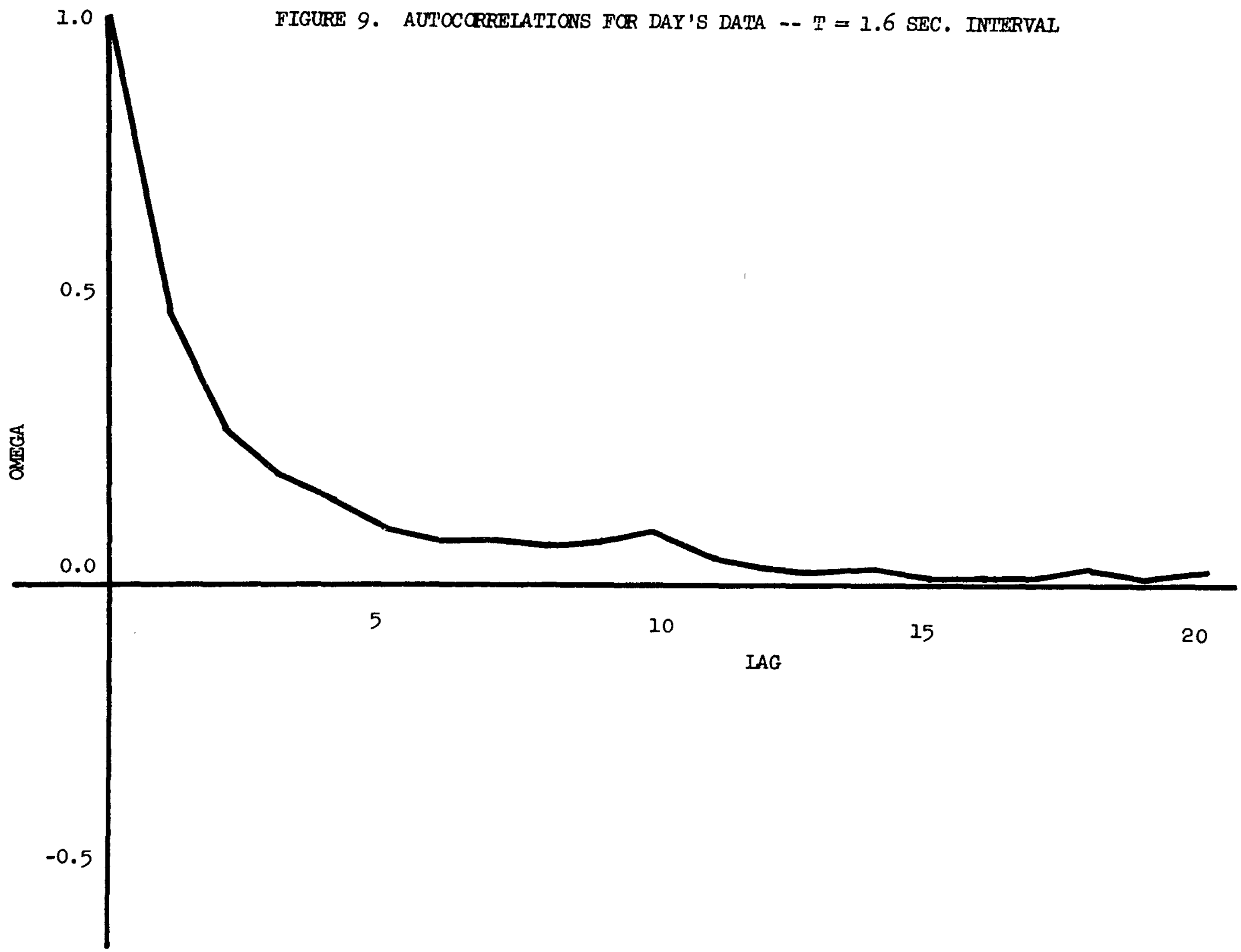
the plots of autocorrelation vs. lag show a steady decline from high positive values for the initial lags down to values not significantly different from zero. All the individual plots show this effect, as well.

2. For the intervals T = 4.2, T = 7.1, and T = 10.6, the autocorrelations vary erratically near the chance level, although this is not uniformly true of all subjects, 5 out of 15 records being found which are similar to those for the shorter intervals, and 2 records showing a marked tendency toward alternating positive and negative autocorrelations.

On the basis of these plots and upon an analysis of "runs" (sequences of identical responses), Day concludes that there is a non-random serial effect for the short inter-stimulus intervals which all but disappears as the interval T is lengthened. This non-random serial effect is distinguished by the presence of ultra-long successions of "yesses" and ultra-long successions of "nos." No attempt is made to explain the phenomenon, which could be indicative of the properties of the auditory threshold or could be an artifact of the experimental design, or something else entirely.

Treating Day's lag correlations as though they were lag correlations based upon a quantified instead of a dichotomous variable, spectra were computed from these lag correlations for all subjects and all values of

FIGURE 9. AUTOCORRELATIONS FOR DAY'S DATA -- T = 1.6 SEC. INTERVAL



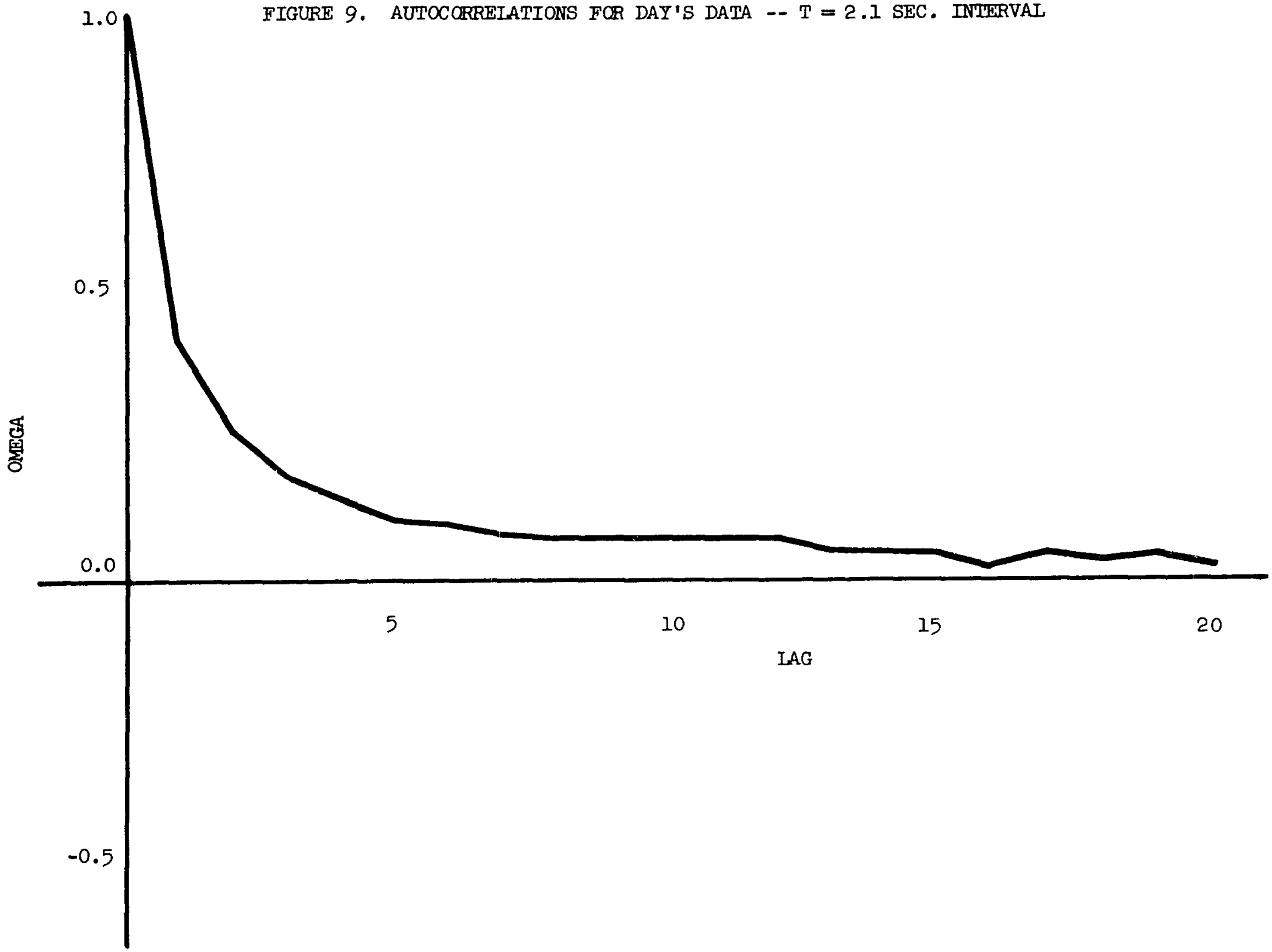
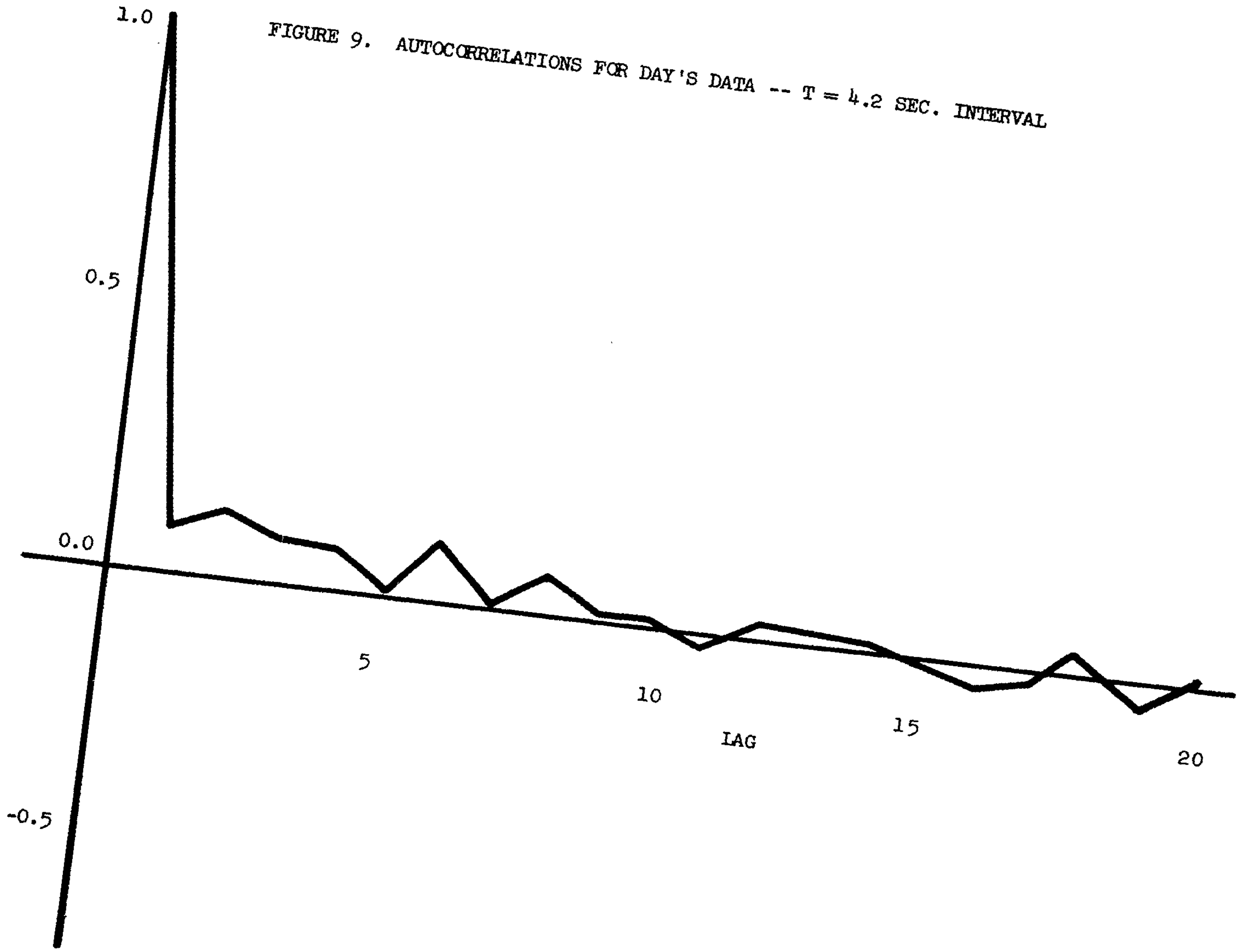


FIGURE 9. AUTOCORRELATIONS FOR DAY'S DATA -- T = 2.1 SEC. INTERVAL

FIGURE 9. AUTOCORRELATIONS FOR DAY'S DATA -- T = 4.2 SEC. INTERVAL



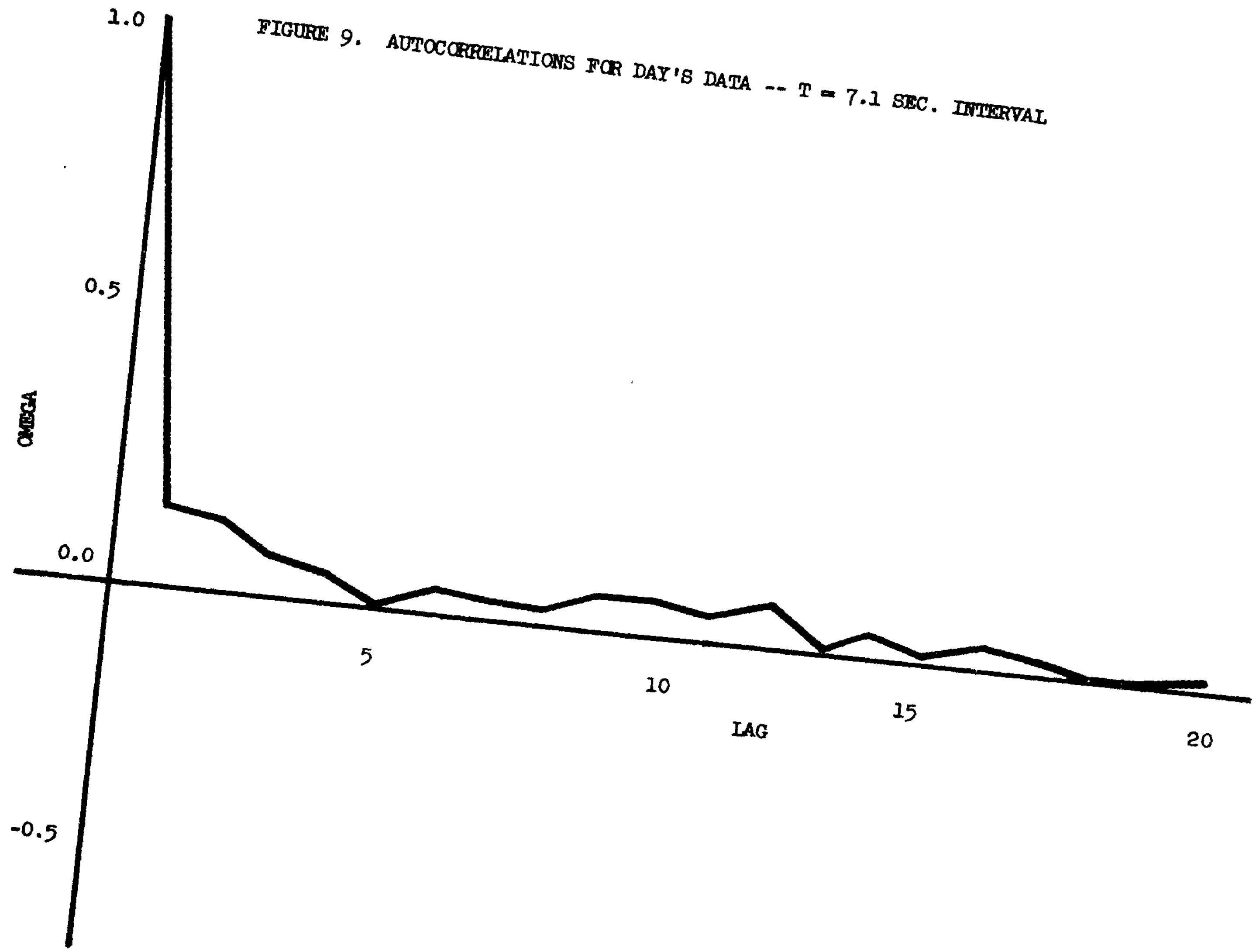


FIGURE 9. AUTOCORRELATIONS FOR DAY'S DATA -- T = 7.1 SEC. INTERVAL

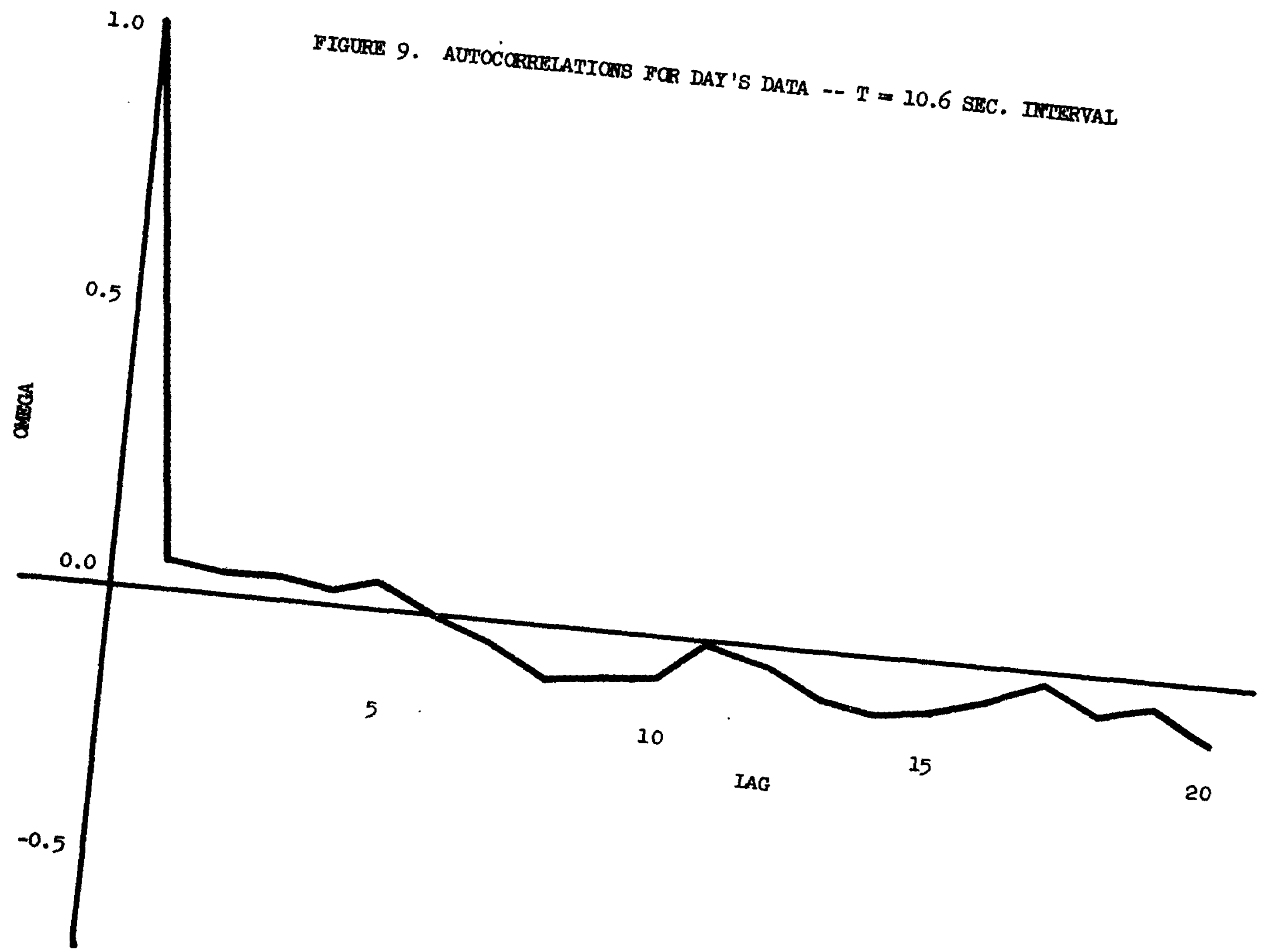


FIGURE 9. AUTOCORRELATIONS FOR DAY'S DATA -- T = 10.6 SEC. INTERVAL

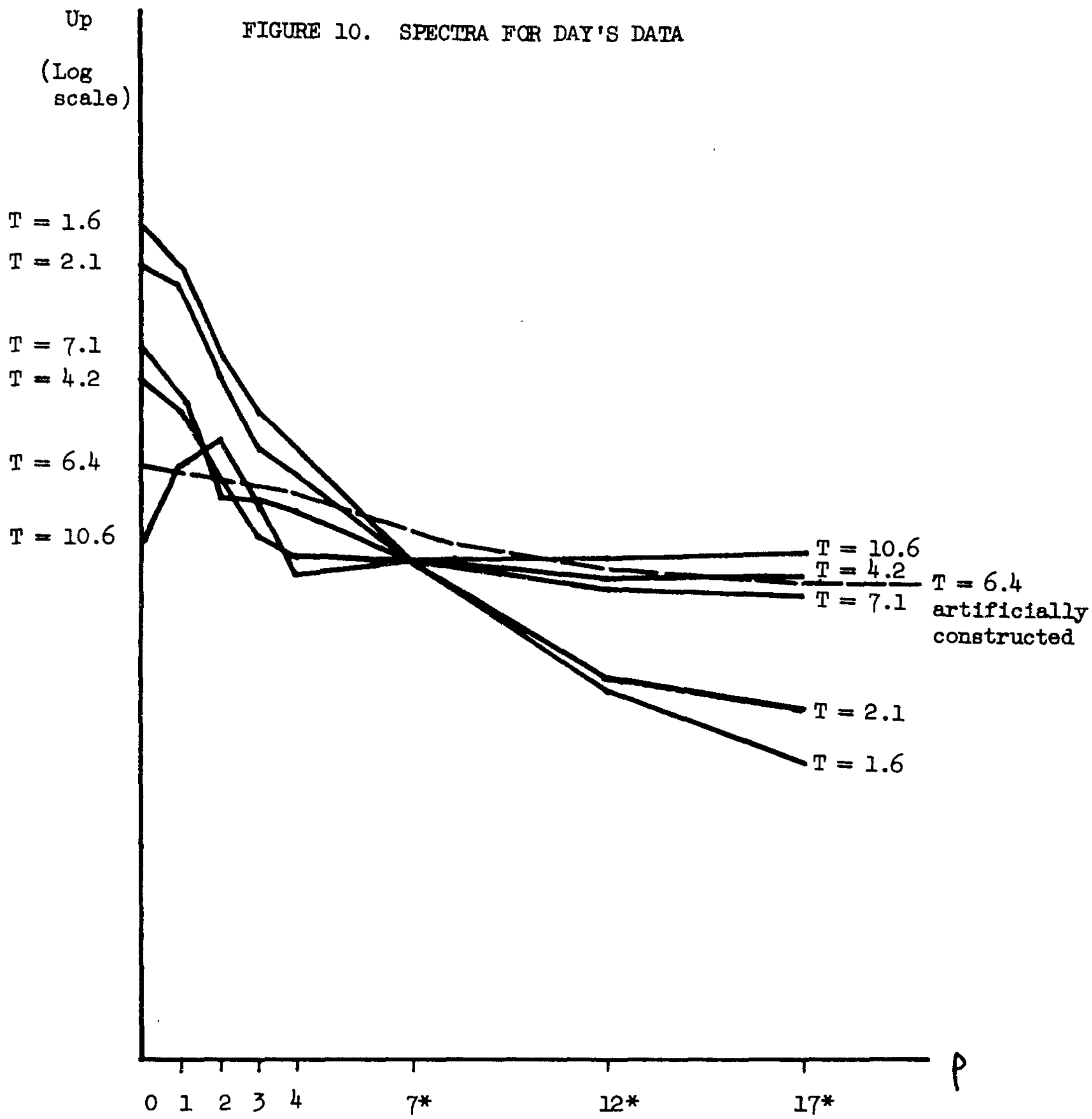
T, the inter-stimulus interval. Our main interest here is with the effect of T and not with individual differences. Accordingly, we content ourselves with noting, as Day did, that individual differences are small for $T = 1.6$ and 2.1 but are much more apparent for $T = 4.2$, 7.1 and 10.6; the spectra presented in Figure 10 for the five values of T each represent averages over the five subjects. The sixth spectrum in the Figure is explained below. We ask two questions which Day did not consider:

1. Do these five spectra all represent a single underlying process which is a function of time alone? That is, do the spectra appear different only because they are each drawn to a different time scale; e.g., for the spectrum with $T = 1.6$ seconds, the point $p = 4$ represents a frequency of 1 cycle per 10 observation points or 1 cycle/16 seconds, whereas for the spectrum with $T = 4.2$ seconds, the point $p = 4$ represents 1 cycle per 10 observation points or 1 cycle/42 seconds.

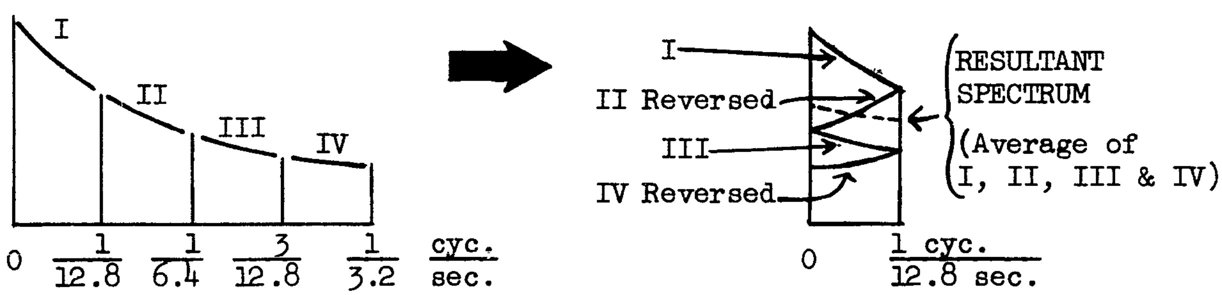
2. Whether the five spectra represent one process or five processes, can we go beyond description in organizing the results? In other words, why is the process (or processes) non-random at all, and further, why does the non-randomness take the particular form that it does.

In order to answer question 1, let us divide the spectrum with $T = 1.6$ seconds into equal quarters along the frequency scale. The first quarter extends over the frequency range 0 to 1 cycle/12.8 seconds; the second quarter extends over the range 1 cycle/12.8 seconds to 1 cycle/6.4 seconds; the third over the range 1 cycle/6.4 seconds to 3 cycles/12.8 seconds, and the fourth over the range 3 cycles/12.8 seconds to 1 cycle/3.2 seconds. Let us construct from these pieces the shape of the spectrum of the same process referred to the base $T = 6.4$ seconds. The entire frequency range of this constructed spectrum would extend from 0 to 1

FIGURE 10. SPECTRA FOR DAY'S DATA



cycle/12.8 seconds; the maximum frequency which would appear in this artificially constructed spectrum is the frequency (1 cycle/12.8 seconds) corresponding to the upper limit of the first quarter of the T = 1.6 second spectrum. Thus the first quarter of the T = 1.6 second spectrum carries over directly to cover the whole range of the constructed spectrum. Frequencies above 1 cycle/12.8 seconds, however, must enter the constructed spectrum as 'aliases' (see p. 20). For instance, the frequency 1 cycle/6.4 seconds appears to be a frequency of 0, since observations are only being made every 6.4 seconds, the precise interval at which a wave of frequency 1 cycle/6.4 seconds returns to its starting point. Referring to the rules for aliasing given in Figure 4B, the artificial spectrum can be constructed from the four pieces of the T = 1.6 second spectrum. (The accompanying diagram shows how each piece carries over; the four pieces are averaged to give the constructed spectrum.)



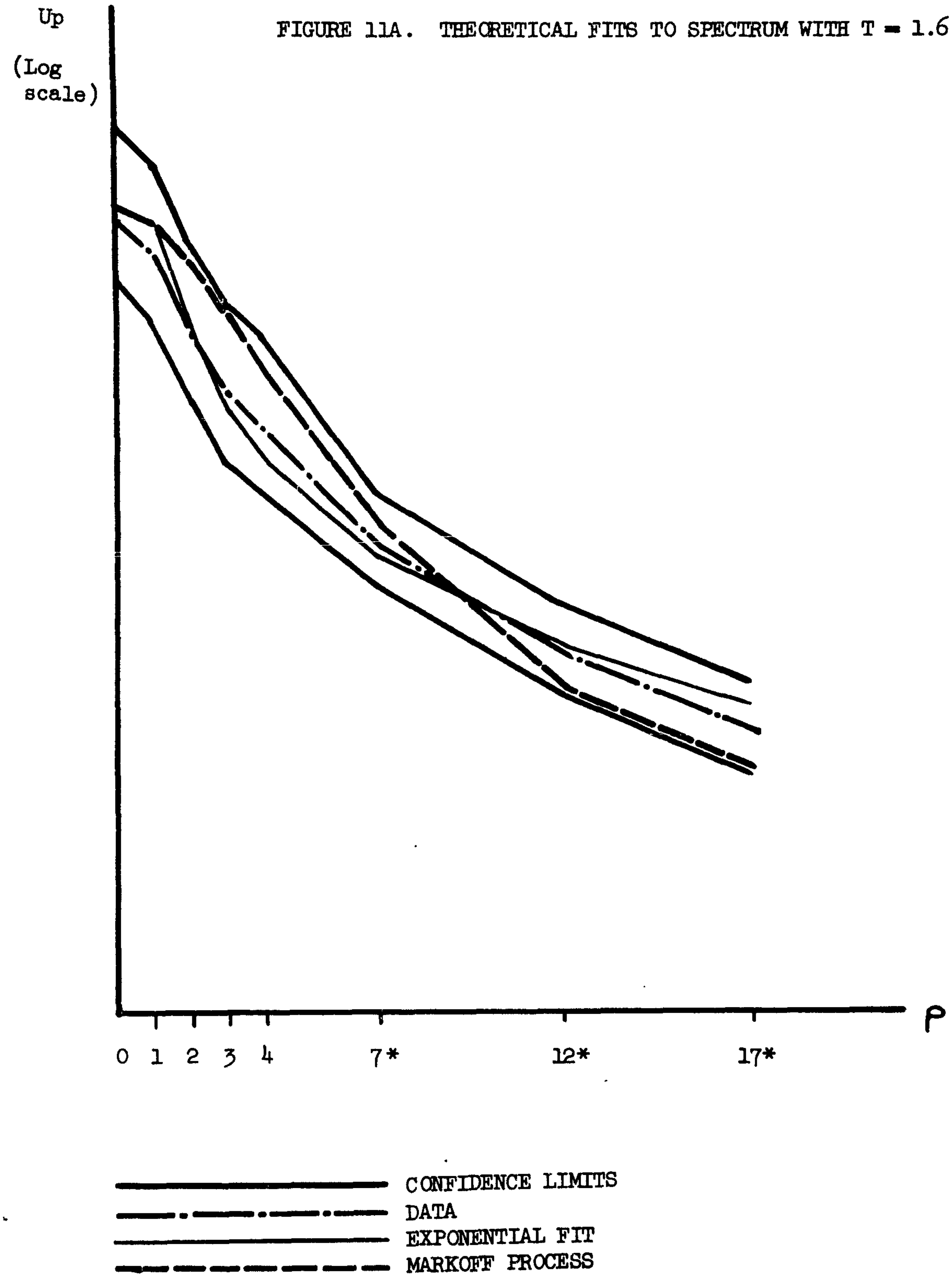
Pieces I and III carry over directly; pieces II and IV are turned backwards, due to the nature of aliasing, and then are carried over.

The artificially constructed spectrum with T = 6.4 seconds is shown along with the empirical spectra of Figure 10. It will be seen that the spectrum for T = 6.4 seconds constructed to represent the <u>same process as the spectrum with T = 1.6 seconds</u> shows less deviation from randomness (i.e., is flatter) than the empirical spectrum with T = 7.1 seconds (at

$p = 0$, the $T = 7.1$ second spectrum is significantly above the $T = 6.4$ spectrum at a significance level of .005). In other words, at the interval $T = 7.1$ seconds, the psychophysical judgment process is more non-random than would be expected if we were dealing here with a process (a fluctuating threshold) that varied passively with time alone. This conclusion is weakened by the weird behavior of the spectrum for $T = 10.6$, due entirely to two subjects who showed a tendency toward alternating "yes" and "no" responses. But if the conclusion is accepted, it can only mean that the threshold is affected by the fact that we are measuring it. A perceived differential increase in the intensity of the sound raises the probability that the increase will be heard upon the next presentation. We do not know whether this facilitative effect is due to afferent or central neural facilitation or to the fact that the subject, <u>having responded</u> (pressed the key) is placed in greater readiness for the next stimulus. But we do know that the spectra we are dealing with are not the spectra of the fluctuations of threshold alone. Some further effect is present.

As to the question of the explanation, or at least greater specificity of classification of the non-randomness at hand, let us consider the spectra for $T = 1.6$ and 2.1 seconds in greater detail (for the longer intervals, individual differences become operative, and the average curves are no longer as smooth nor as meaningful). In Figures 11a and 11b are shown two theoretical fits to each of these two empirical spectra along with the 95% confidence belts which surround each of the empirical spectra. As can be seen from the Figure, both fits are quite adequate to the data, and either one might serve as a possible explanation of the phenomena.

FIGURE 11A. THEORETICAL FITS TO SPECTRUM WITH T = 1.6 SECS.



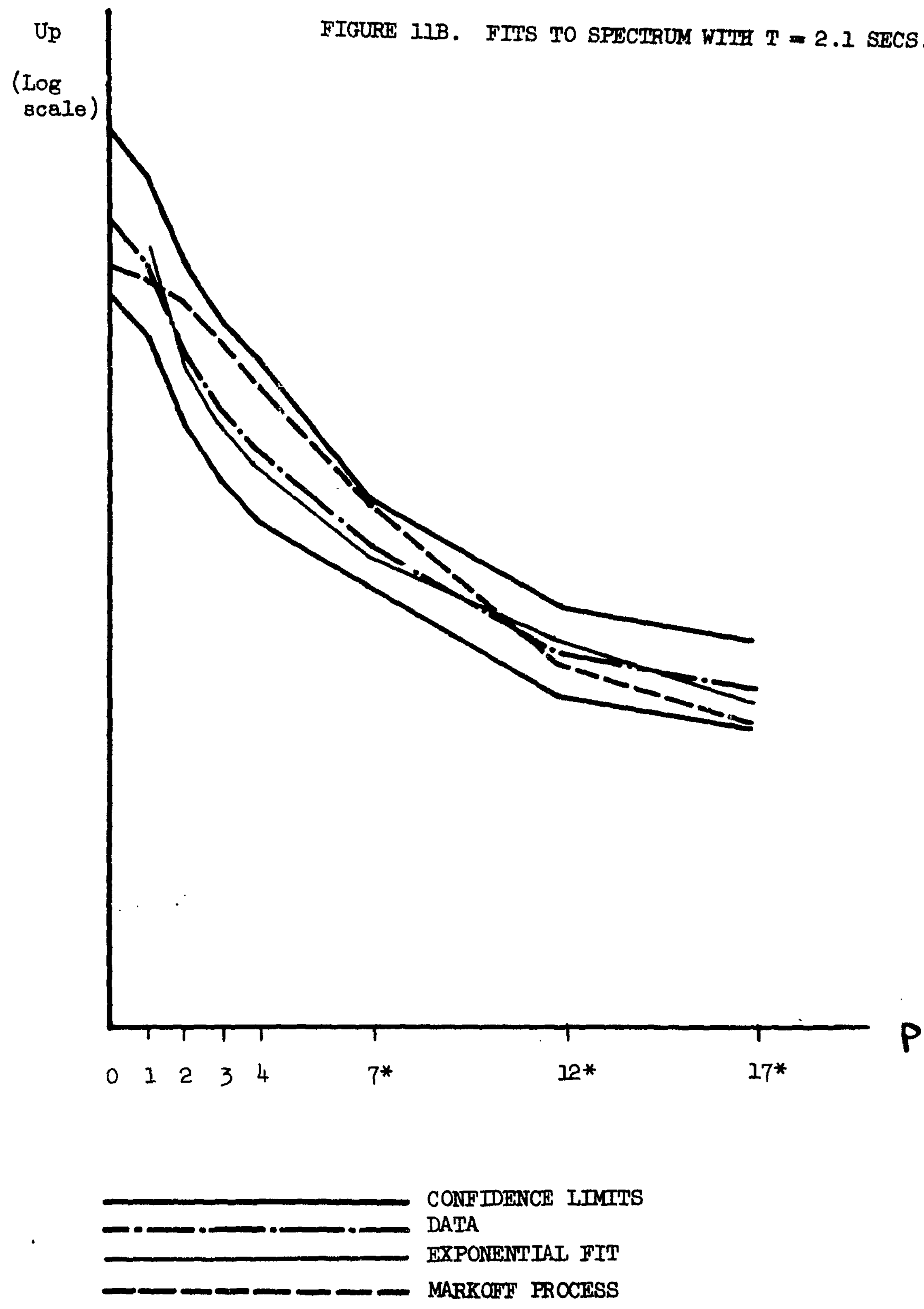


FIGURE 11B. FITS TO SPECTRUM WITH T = 2.1 SECS.

One of these fits is according to the relation $\log U_p = A - B \log p$. This fit is drawn out of a hat, with no theoretical justification. It was merely noticed that the spectra (for T = 1.6 seconds and T = 2.1 seconds) when plotted on log-log paper instead of semi-log paper, produced relatively straight lines. (The value p = 0 is omitted, since the log (0) is negatively infinite.) Perfect straight lines were fitted to the data on the log-log plots by eye and the values transferred back to the semi-log plot. The fit to the data is excellent; the author knows of no model, however, which would produce such a spectrum.

The second fit was found from the following theoretical model: Suppose that the probability of responding "yes" to any given stimulus (i.e., hearing it) depends only upon whether the <u>previous</u> stimulus was heard or not heard. Suppose that the probability of a "no" response likewise depends only upon the nature of the previous response. Then a table can be constructed of the conditional probabilities of: a 0 (no) followed by a 1 (yes), a 0 followed by a 0, a 1 followed by 1, and a 1 followed by a 0.

being followed by:

Probability of:		0	1
	0	P	Q
	1	Q	P

(P is some probability between 0 and 1; Q = 1 - P, since a 0 must be followed by <u>something</u>; the probability of a 0 followed by a 0 equals the probability of a 1 followed by a 1 since the number of 0's is to equal the number of 1's in the entire series.)

This model is a special case of a "Markoff process" (cf. [13]).

The lag covariances for the process can be computed from (5.3) which states that $Q_j = \frac{A_j}{N} - \frac{1}{4}$, where A_j is the number of matching pairs of 1's at lag j. Evidently A_j is equivalent to the total number of 1's in the entire series times the probability P_j that, if a 1 appears, another 1 will appear j responses later. But the number of 1's in the entire series is $\frac{N}{2}$. Thus

$$(5.6) \qquad Q_j = \tfrac{1}{2} P_j - \tfrac{1}{4} = \tfrac{1}{4}(2P_j - 1), \text{ where } \begin{array}{l} P_1 = P \\ P_0 = 1 \end{array} \quad .$$

The P_j can be related to the parameter P by enumerating the sequences of 0's and 1's which can separate two 1's by j responses, calculating the probabilities of each, and adding. When this is done it turns out that

$$(5.7) \qquad Q_j = \tfrac{1}{4}(2P - 1)^j \quad .$$

Choosing for T = 1.6 the value P = .80 and for T = 2.1 the value P = .75, the Q_j's were computed by (5.7) and thence the U_p's were computed (from formula (4.4)). Other values of P in the same general neighborhood would do approximately as well; the particular values chosen are only for illustrative purposes.

In summary, the empirical spectra for the short inter-stimulus intervals in Day's experiment can be closely approximated by either of two fits; one of these fits has no theoretical justification, although one might conceivably be found. The other results from the hypothesis that the probability of a stimulus being heard depends only upon whether the

previous stimulus was heard or was not heard, being high (P) in the former instance and low (Q) in the latter instance. This facilitative effect is still present at longer inter-stimulus intervals in much lesser degree. Day's experiment cannot be considered to have measured directly a passively varying threshold, since the spectra for the short and long inter-stimulus intervals are not commensurate with each other. Spectral analysis has considerably sharpened the conclusions from Day's experiment, although several questions remain unanswered.

Spectral analysis having passed these two preliminary tests satisfactorily, we are now ready to apply it to the experiment on individual differences in performance on routine, repetitive tasks.

VI

EXPERIMENTAL DESIGN

In Section I, the following hypotheses were suggested as being subject to experimental test:

1. Reliable individual differences in the performance of repetitive tasks can be discovered by objective analysis of long series of experimental observations which are ordered along the time dimension.

2. The differences discovered by objective analysis in one task situation are of a general nature, i.e., certain stable features of personality are manifested on all repetitive, routine tasks. The concepts to which these personality features might be allied are: rigidity, level of aspiration, involvement, concern over errors, stability and attention.

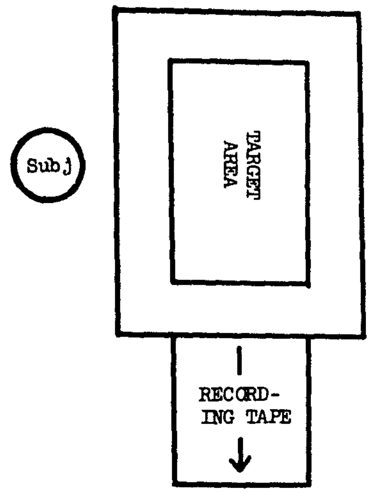
The discussion of Sections III, IV and V has led to the decision to use spectral analysis as a mathematical method in dealing with time-ordered observations. The choice of an experimental task is subject to several considerations: 1) A perceptual-motor performance task is more suitable than a task which is predominantly perceptual or predominantly motor or predominantly mental, since it is more likely to involve the whole organism and to avoid awkward breaks in the activity; 2) it is desirable to minimize the factors of learning and fatigue, since they introduce too many uncontrollable variables, and complicate the mathematical analysis enormously by making the time series non-stationary; 3) a long series of observations (100 or more) is necessary for statistical reliability; accordingly, the task has to be one in which the rate of production of observations is rapid; 4) the observations should be easily measurable

and in convenient form for analysis; 5) the goal of the perceptual-motor task should be kept constant, for otherwise the results continge too heavily upon the manner in which the goal changes,[7] an extraneous and complicating experimental variable; 6) on the other hand, the task must not be so easy that no variability is manifest; in other words, errors should be unavoidable; 7) insofar as is possible, skill factors and other idiosyncratic factors not strictly psychological should not enter importantly; 8) there should be a potential versatility in the relevant experimental conditions.

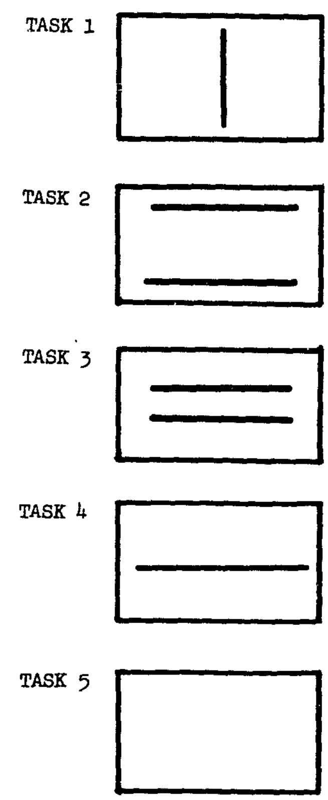
In the light of all these considerations and a number of pilot studies, the task finally chosen was that of jabbing a stylus at a target. The subject stood alongside a table on which a rectangular box was placed so that its long side was coincident with the edge of the table. A square of paper atop the box was designated as the target area. (Figure 12A.) The specific goal within the area was indicated by a black line or lines drawn on this square of paper. Below the paper was a metal backing covered with strips of typewriter ribbon. Between the paper and the ribbon-covered backing was a tape which moved at the constant rate of one inch a second across the length of the box (from left to right) and

[7] This is the difficulty with the study by Krendel (24) referred to previously as a case in which spectral analysis was applied to data of a psychological nature. He used a pursuit tracking task in which the subjects tried to return a continually moving spot of light to a central position on a screen by means of a rudder control. The spectrum of the position of the rudder was found to conform closely to the spectrum of the position of the moving spot. From an engineering standpoint, this is important, but for psychological purposes it is disappointing, since it is the deviations from optimal operation which are psychologically the most interesting (and most subject to individual differences). These deviations are obscured by the large effect of conformity of the subject's activity to the demands of the varying goal.

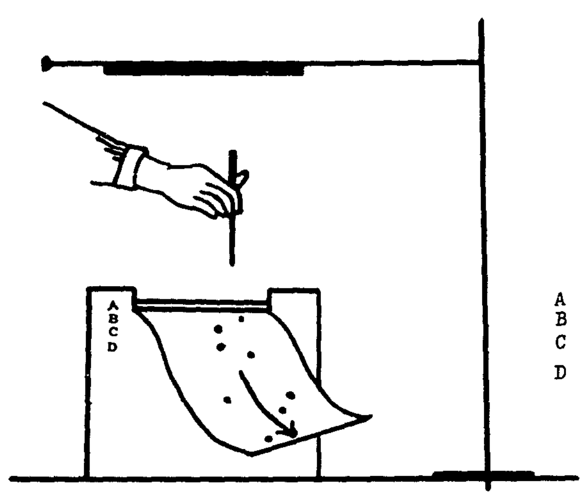
12A TOP VIEW



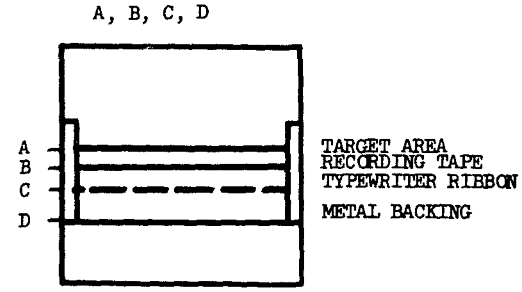
TARGET AREAS



12B SIDE VIEW



DETAIL OF SIDE VIEW
A, B, C, D



registered a black dot wherever the subject jabbed (Figure 12B). The impact of the stylus was sufficient to leave an indentation on the target paper. To minimize the effect of the subject being influenced by the knowledge of where his previous jabs had landed, the target papers were prepared beforehand so as to be already covered with a large number of indentations. The subject gripped the stylus in the middle, with the top of the stylus touching a backstop placed fifteen inches above the target area. When held in this position, the point of the stylus was eight inches from the target area. After each jab at the target area, the subject lifted the stylus back up until contact with the backstop was made, and jabbed again. The subject jabbed at his own rate and continued until he had produced 100 jabs for each of five task variations.

The variations in task were achieved by changing the paper square comprising the target area. Each of five paper squares had a different goal indicated on it, namely:

1. A wide black band extending away from the subject vertically; i.e., perpendicular to the direction of motion of the recording tape.

2. The area between two thin <u>horizontal</u> lines spaced respectively 26/50 inches and 74/50 inches from the edge of the box nearest the subject.

3. The area between two thin horizontal lines spaced respectively 42/50 inches and 63/50 inches from the edge of the box nearest the subject.

4. A single thin horizontal line spaced 45/50 inches from the edge of the box nearest the subject.

5. A blank paper.

The five tasks were presented to the subjects in the same order that they have been described above. The following instructions were given before the presentation of each task:

Task 1. "Aim for the thick black line. Hit the line as often as you can. Jab the stylus in a quick motion--don't 'sneak up' on the line. Make sure you touch this backstop with the top of the stylus before each jab."

Task 2. "Aim between the two lines, anywhere between them."

Task 3. "The last series was just a warm-up for this next one, where we've made things a little harder for you. Aim anywhere between the two lines."

Task 4. "Now aim for this single line."

Task 5. "I am going to replace this paper by a paper with no line drawn upon it. Aim for the same spot that you have been aiming at. Remember the position of the line relative to the box and, after I have changed the paper, pretend that the same line is still in the same place. Got it?"

The subjects were allowed a few practice jabs before the beginning of the first task to get the feel of the activity. Any confusion over the instructions was resolved before the performance began. For each task, the experimenter silently counted 100 jabs and then told the subject to stop. The entire succession of five tasks took a total of approximately five minutes to perform. The dots on the recording tape gave a record of the distance of each jab from the edge of the box nearest the subject. (See Figure 12B.) This was the time-series measurement. The serial order of occurrence of the jabs was given directly by the tape.

Task 1 measured whether the subject jabbed at the top of the line or at the bottom of the line and constituted an attempt to see if there were interesting individual differences in the way the subjects sought variety in the task (since the place along the line where the subject jabbed was irrelevant to the requirement that the line be struck somewhere). Tasks 2, 3, 4--where the lines demarking the goal area were widely separated, moderately separated, and finally collapsed into one line--made up a graded series from almost complete objective success to almost complete objective failure, and were intended to be increasingly difficult, involving and stressful. Task 5, wherein the objective standard was removed, was employed in the hope that those subjects who were dependent upon the presence of an objective standard would show a deterioration in performance.

Task 1, then, was relevant to rigidity;[8] Tasks 2, 3 and 4 to level of aspiration, involvement, and concern over errors; Task 5 to stability in the face of changing external circumstances, and all the tasks to attention. The general assumption which related these effects to the objective framework of spectral analysis was that the higher the level of aspiration, the involvement, the concern over errors, and/or the attention, the greater would be the tendency to attempt to compensate for errors. This would result in a time-series showing sharp, quick ups and downs, and thus a spectrum containing a preponderance of high frequency components. On the other hand, a low level of aspiration, low involvement, little concern over errors and/or inattention would tend to produce

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[8]"Rigidity" as used here connotes the unwillingness to explore differing available means of performance of an activity.

a time series of a slow wandering character, and thus a spectrum with a preponderance of <u>low frequency components</u>. Either instability or lack of rigidity (i.e., either inability to maintain constancy of performance on Task 5 or willingness to be variable on Task 1), meanwhile, would be expected to increase the variance of the time-series measurements. Motor skill undoubtedly enters to some extent in the jabbing activity and would be expected to decrease the variance.

The subjects were thirty-three Princeton undergraduates, all of whom are members of a special student group being studied by the Study of Education at Princeton Project under the direction of Professor Frederick Stephan. All these subjects were tested in March, 1952. Fifteen of the subjects were retested on the same five tasks a month later.

VII

EXPERIMENTAL RESULTS: MEASURES OF INDIVIDUAL DIFFERENCE

Statement of Results

For each of the five experimental tasks described on page 42, three sources of individual differences are available: the spectral densities U_p, the variance, σ^2, of the observations about the mean observation, and the speed, s, with which the subject produces observations (in jabs/second). Measures other than these have not been analyzed by reason of being less fundamental from an analytic standpoint (i.e., the mean observation, the percentage of "successes" as defined by the task, etc.).

The spectral densities were computed from the first 20 lag correlations by IBM punch-card procedures explained in detail in Appendix C. The values of U_p for p = 0, 1, 2, 3, 4, 7,* 12* and 17* are presented in Table 1 for each subject and task for test and retest. The variances were a by-product of the computation of the U_p. They are given in Table 2. The speed of jabbing was easily measured from the total length of tape needed to record 100 jabs, since the tape moved at a constant rate of one inch a second. The values of s appear in Table 3. These data were first analyzed for reliability. The reliabilities of the measures σ^2 and s, task by task, (based on the fifteen subjects, number 1 - number 15, for whom retest data was available) are as follows:

	σ^2	s
Task 1	.159	.580
Task 2	.422	.833
Task 3	.379	.828
Task 4	.450	.830
Task 5	.463	.840

TABLE 1

SPECTRAL DENSITIES, U_p, FOR EACH TASK, SUBJECT, AND VALUE OF p

TASK 1, TEST

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	48257	27861	4871	1072	1065	856	911	608
2	2046	1650	1700	1967	1355	838	955	640
3	81830	50069	12810	6364	4922	2988	1336	1380
4	6556	3854	783	300	603	498	710	608
5	36572	24849	12954	8378	3120	889	477	572
6	38981	37197	29409	16129	10342	5940	1421	712
7	6960	6474	4456	2484	1679	874	646	651
8	4310	5046	5228	3915	2680	831	903	591
9	19520	15488	7512	4072	5615	5402	3119	3356
10	19721	14459	7960	4940	2752	1291	1121	3363
11	109326	64533	13181	3008	813	965	855	1097
12	31806	19583	4652	1038	653	530	490	631
13	1825	2270	2846	3597	3909	1611	615	869
14	54816	40233	15167	3799	2326	679	9	505
15	6910	5999	3432	1296	951	1153	383	575
16	11345	7451	3127	2661	2654	1423	559	701
17	24143	28968	41884	54232	46372	11006	2564	1952
18	1014	909	748	537	331	535	498	506
19	12734	7689	2284	1884	1595	829	893	680
20	1462	1493	1009	563	821	886	447	667
21	5672	5803	5249	3304	1631	1891	1626	1255
22	5850	4905	4193	4233	3906	1392	657	384
23	10803	8451	4772	2321	984	1014	1102	1860
24	824	863	716	444	295	476	753	963
25	20091	11754	2271	724	695	373	286	305
26	12772	8994	4227	2572	2246	3104	2918	2518
27	57966	41575	15177	3688	3169	1824	950	2156
28	13191	8030	2365	1629	1713	1247	930	860
29	4266	2578	979	1264	1374	770	430	563
30	26612	37301	38030	21436	11631	2954	1350	1099
31	18215	15145	9094	4677	3223	2150	473	869
32	40956	29320	11942	4974	2861	689	9	609
33	37670	26029	11070	5047	882	911	106	1147

(For a random series with $\sigma^2 = 10$, all values of U = 196.)

TABLE 1

SPECTRAL DENSITIES, U_p, FOR EACH TASK, SUBJECT, AND VALUE OF p

TASK 2, TEST

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	1134	872	596	712	732	694	913	745
2	472	733	830	496	372	555	674	1184
3	935	1221	1292	816	668	1196	1456	904
4	457	327	202	310	448	528	639	955
5	195	347	755	1098	1035	735	823	668
6	1641	1848	2057	1845	1495	1930	2000	1983
7	2008	1606	1229	1999	2051	957	1365	753
8	853	1047	1020	662	647	614	802	819
9	1485	2368	3536	3848	3281	2029	2464	3406
10	768	803	738	568	627	710	1556	1379
11	97	328	970	1191	804	638	831	717
12	527	634	610	435	481	642	909	687
13	840	797	568	432	658	677	907	1180
14	403	338	318	385	362	789	550	799
15	6566	4481	2216	1870	1191	352	434	1235
16	777	824	676	420	327	406	833	653
17	580	1160	1429	807	561	1087	1711	2029
18	174	225	211	159	236	374	773	549
19	866	967	999	975	1192	1255	1877	1639
20	2264	1704	1006	683	466	469	608	506
21	1489	1167	657	466	393	608	618	818
22	1440	1597	1526	1410	1626	747	457	383
23	1709	1384	962	990	889	753	875	1438
24	792	783	653	480	419	562	765	580
25	1888	1588	943	619	514	577	371	480
26	1616	1423	1119	920	1081	2147	959	1207
27	1401	1090	594	432	407	625	1519	1114
28	1561	1500	1706	1636	1042	1356	376	830
29	1875	1677	1306	990	584	695	278	304
30	197	307	727	981	662	1253	877	831
31	424	482	636	984	1095	778	744	816
32	4058	3020	1369	1036	1072	1082	1083	1176
33	687	723	715	572	559	675	773	462

(For a random series with $\sigma^2 = 10$, all values of U = 196.)

TABLE 1

SPECTRAL DENSITIES, U_p, FOR EACH SUBJECT, TASK, AND VALUE OF p

TASK 3, TEST

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	733	855	1129	914	349	621	533	778
2	772	691	517	345	211	452	469	873
3	555	590	618	500	322	704	933	1297
4	233	167	80	137	206	361	624	300
5	890	665	335	362	494	701	885	538
6	839	788	848	933	831	1208	1080	973
7	539	418	387	608	994	849	436	268
8	576	417	254	266	331	553	485	499
9	1871	1904	1718	1266	968	1396	2177	2192
10	651	642	524	397	440	675	1284	1319
11	530	461	370	472	557	686	1398	949
12	101	190	301	388	547	522	436	307
13	610	570	521	587	602	452	410	723
14	337	270	181	172	163	304	283	251
15	918	1259	1535	1104	875	614	880	605
16	405	451	546	507	561	877	407	494
17	408	543	883	1169	1108	1290	1771	1070
18	640	632	529	347	313	440	483	613
19	300	355	495	551	463	781	369	698
20	2901	2355	1643	988	521	498	311	324
21	678	752	765	828	958	772	493	651
22	510	582	656	665	644	914	658	238
23	809	589	271	205	262	518	784	595
24	292	233	124	53	69	214	557	390
25	984	735	455	462	492	496	356	712
26	1377	889	425	583	772	813	665	813
27	805	632	388	322	418	612	771	585
28	942	706	423	472	569	610	797	659
29	322	280	172	168	381	276	247	212
30	472	535	736	1103	1435	816	401	785
31	346	295	432	745	778	529	1024	612
32	1843	1341	633	269	177	617	815	1258
33	169	367	622	643	591	741	928	668

(For a random series with $\sigma^2 = 10$, all values of U = 196.)

TABLE 1

SPECTRAL DENSITIES, U_p, FOR EACH SUBJECT, TASK, AND VALUE OF p

TASK 4, TEST

SUB-JECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	1952	1582	1508	1666	1113	623	890	918
2	773	549	347	368	275	301	841	532
3	433	611	815	761	685	913	784	704
4	493	430	396	419	363	681	435	650
5	272	362	431	322	153	372	733	552
6	2459	2205	1843	1317	720	1809	1277	846
7	328	259	215	281	355	921	574	307
8	706	722	907	1004	663	423	638	285
9	2254	1823	1042	811	1105	1642	1095	1407
10	227	183	257	572	870	961	783	1272
11	147	198	371	637	666	1000	1588	832
12	141	162	332	492	387	455	340	191
13	216	199	190	198	201	359	303	298
14	195	259	336	327	279	259	276	193
15	4561	3299	1377	694	1011	923	556	980
16	513	563	620	593	442	713	869	566
17	1208	1125	781	692	975	879	1016	1546
18	143	138	164	222	320	581	719	478
19	363	368	355	327	342	561	672	433
20	975	1028	901	718	580	464	729	471
21	313	386	419	486	717	642	970	995
22	723	778	964	766	394	718	454	689
23	22	60	186	335	297	293	743	288
24	139	283	575	651	492	515	406	258
25	307	289	347	483	531	456	332	306
26	709	701	572	316	181	631	514	671
27	192	326	605	614	467	486	712	715
28	1299	1176	903	788	651	614	387	590
29	580	497	406	411	291	179	176	180
30	454	467	962	2018	2568	1410	648	530
31	219	261	221	193	310	564	558	596
32	806	691	409	258	395	666	996	916
33	94	152	212	215	196	323	421	481

(For a random series with $\sigma^2 = 10$, all values of U = 196.)

TABLE 1

SPECTRAL DENSITIES, U_p, FOR EACH SUBJECT, TASK, AND VALUE OF p

TASK 5, TEST

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	3853	2659	907	599	1570	1029	1077	607
2	963	1133	1028	628	385	534	926	686
3	737	711	762	736	617	1300	1070	1319
4	2950	1758	434	206	210	334	737	667
5	395	538	575	477	535	892	760	684
6	3431	2491	1723	1655	1235	1408	1813	898
7	2978	2030	743	410	370	325	191	279
8	1792	1413	727	410	342	335	592	200
9	687	926	1005	1203	1848	1264	746	964
10	1081	1036	898	787	730	667	1091	1095
11	1728	1088	446	484	782	1308	735	608
12	715	463	193	158	173	159	201	178
13	311	448	532	504	579	424	505	836
14	462	374	233	239	359	318	288	172
15	3688	3485	2536	1307	804	687	859	823
16	662	484	317	577	877	721	474	675
17	500	637	774	821	631	1015	1033	822
18	646	685	609	446	324	366	723	405
19	782	779	856	957	774	570	483	708
20	1730	1463	1035	762	969	685	434	468
21	14492	11314	6276	3793	3035	1587	1108	789
22	401	398	445	437	320	267	374	365
23	366	336	241	213	383	480	730	377
24	595	410	244	432	613	406	645	313
25	496	498	439	377	305	207	327	299
26	1472	1095	705	708	700	687	638	840
27	714	748	697	481	368	402	792	904
28	893	691	476	470	513	962	921	466
29	1832	1365	608	259	356	347	313	234
30	2834	2454	1554	880	823	857	482	593
31	1302	834	407	535	874	608	658	633
32	944	1217	1566	1560	1099	879	1260	392
33	730	668	510	363	285	641	466	216

(For a random series with $\sigma^2 = 10$, all values of U = 196.)

TABLE 1

SPECTRAL DENSITIES, U_p, FOR EACH SUBJECT, TASK, AND VALUE OF p

TASK 1, RETEST

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	24782	18185	7484	3635	3198	1143	967	1624
2	3615	2425	1001	644	511	269	196	356
3	5951	3988	1664	1183	1875	1926	1412	556
4	1802	1789	1527	972	502	470	717	689
5	7301	6704	5860	3780	1497	1372	695	824
6	96863	63425	22702	12842	7724	1618	656	732
7	3822	2717	1240	729	559	1023	673	463
8	23255	16743	6686	2282	1278	1360	972	535
9	5741	6504	5994	5072	6054	3348	2415	2622
10	4849	3892	2028	815	414	664	1061	1189
11	1394	1593	1443	1131	1071	1040	882	701
12	38264	24793	7671	2120	629	471	401	239
13	594	564	503	676	699	592	512	754
14	560	696	835	1009	1065	583	771	668
15	15454	9631	3280	2207	1767	1274	603	416

TASK 2, RETEST

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	3625	3105	2032	1325	1141	1335	1326	1098
2	629	565	344	182	296	307	494	409
3	5579	4038	2433	2767	3035	1902	1463	1602
4	525	601	594	414	382	627	354	755
5	543	578	432	350	578	928	696	795
6	367	1081	2793	3505	3286	1124	964	1452
7	1176	1323	1711	1710	1108	1193	535	736
8	7350	5061	1966	777	719	1084	736	678
9	830	814	836	935	1012	1726	1583	1059
10	2997	2553	1500	1004	1109	591	658	1188
11	843	916	649	480	1129	2041	837	959
12	1826	1292	555	493	891	920	719	548
13	1154	947	670	585	812	501	658	1091
14	414	460	621	705	581	600	1032	502
15	1076	963	818	964	1416	1131	802	764

TABLE 1

SPECTRAL DENSITIES, U_p, FOR EACH SUBJECT, TASK, AND VALUE OF p

TASK 3, RETEST

SUB-JECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	1589	1219	788	554	292	435	785	1345
2	1504	1044	363	208	305	275	254	369
3	1103	1181	1024	655	562	1085	1191	1256
4	411	282	233	361	351	772	771	841
5	792	552	356	601	1436	1204	994	700
6	1392	1574	1290	1063	2090	2668	1686	1053
7	789	675	539	444	350	933	591	772
8	1790	1194	417	307	490	994	685	606
9	544	541	617	703	651	532	980	655
10	1424	989	501	624	966	1164	938	865
11	452	546	868	1130	1221	1652	586	270
12	303	335	390	317	183	641	410	200
13	259	307	248	212	307	378	865	705
14	539	394	218	264	366	358	275	480
15	1751	1308	659	446	748	1439	938	530

TASK 4, RETEST

SUB-JECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	632	721	1222	1392	926	997	826	717
2	294	333	262	193	266	469	369	162
3	751	687	679	644	532	1211	1016	1652
4	344	342	316	249	295	732	598	295
5	541	644	745	617	698	917	883	643
6	688	758	673	487	529	1067	895	815
7	428	419	416	493	703	945	621	259
8	1532	1175	653	465	508	1032	899	482
9	108	216	480	637	604	785	1049	405
10	118	129	212	313	287	464	708	1544
11	400	418	382	427	835	1486	913	532
12	1247	950	490	511	736	725	747	243
13	181	264	316	288	439	854	587	448
14	472	395	366	453	475	522	480	586
15	16405	10890	4303	2600	2000	3097	1634	895

TABLE 1

SPECTRAL DENSITIES, U_p, FOR EACH SUBJECT, TASK, AND VALUE OF p

TASK 5, RETEST

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}
1	1390	1105	720	544	293	747	547	1138
2	74	72	127	291	403	515	456	159
3	355	362	435	573	761	1672	1047	629
4	568	388	240	266	241	358	268	415
5	562	963	1421	1111	701	872	1094	700
6	1416	1490	1687	1797	1949	1362	1046	686
7	3401	2302	1098	946	1236	1577	873	1281
8	1214	1300	1161	594	530	1077	1001	989
9	1026	1121	1172	996	777	740	940	1087
10	1974	1419	710	469	388	464	411	490
11	184	298	558	761	757	642	639	858
12	1077	757	421	404	424	404	783	287
13	660	484	276	328	488	459	592	511
14	1482	956	370	303	241	577	442	647
15	7597	4658	1440	1013	1152	780	871	1323

(For a random series with $\sigma^2 = 10$, all values of $U = 196$.)

TABLE 2

VALUES OF σ^2 (IN [FIFTIETHS-OF-AN-INCH]^2), TASK BY TASK

	TEST					RETEST				
SUBJECT	TASK 1	TASK 2	TASK 3	TASK 4	TASK 5	TASK 1	TASK 2	TASK 3	TASK 4	TASK 5
1	181.1	39.7	34.5	49.1	54.7	162.3	73.0	44.2	46.2	42.0
2	51.0	38.5	29.8	26.7	37.1	27.2	20.3	18.6	16.2	16.9
3	367.5	57.9	44.1	39.4	59.2	80.5	103.7	56.3	58.8	48.8
4	46.8	31.4	18.6	28.1	32.9	39.1	28.2	35.2	24.5	17.1
5	198.1	37.5	33.6	24.8	36.6	92.1	37.6	45.7	39.3	46.4
6	391.1	102.2	52.6	70.4	75.9	435.0	73.6	87.5	43.7	59.6
7	75.3	59.9	27.3	26.4	23.6	46.0	49.5	32.6	26.6	67.7
8	80.1	39.2	24.3	26.7	24.4	135.5	64.1	38.1	40.6	52.0
9	265.1	139.6	95.0	70.3	52.5	175.2	69.3	35.4	34.2	48.2
10	182.1	56.3	48.8	46.5	47.2	63.6	52.0	48.1	39.0	28.2
11	641.4	37.0	45.1	50.4	43.4	49.1	58.9	42.4	43.9	34.2
12	129.0	34.9	20.1	16.4	10.5	153.4	38.8	19.7	30.6	26.0
13	74.8	44.0	27.3	14.9	28.6	32.3	38.4	28.8	28.3	25.4
14	242.1	32.1	13.6	12.7	14.1	36.6	34.2	19.2	25.6	28.6
15	65.7	59.4	40.9	54.6	56.5	92.8	47.7	47.6	144.8	69.8
16	90.6	31.3	28.8	35.0	31.9					
17	1114.8	77.9	64.3	55.7	45.1					
18	28.6	24.5	25.4	25.1	25.6					
19	83.0	74.2	29.5	26.1	32.6					
20	37.9	33.4	32.5	31.6	33.8					
21	110.1	35.7	34.6	39.4	125.9					
22	82.5	38.1	30.5	33.3	17.7					
23	113.0	52.6	29.3	19.6	24.0					
24	36.4	31.7	16.7	20.6	22.9					
25	77.4	31.2	28.0	19.0	15.6					
26	174.8	70.3	39.3	29.1	38.1					
27	303.3	50.6	31.1	31.0	34.2					
28	92.0	51.0	34.0	32.0	37.0					
29	44.4	30.8	12.4	11.9	20.7					
30	380.3	45.8	36.6	50.8	43.8					
31	149.1	39.6	34.4	26.2	33.6					
32	194.3	66.8	45.1	39.8	47.7					
33	185.0	32.1	36.8	18.0	22.6					

TABLE 3

VALUES OF s (IN JABS/SECOND), TASK BY TASK

SUBJECT	TEST					RETEST				
	TASK 1	TASK 2	TASK 3	TASK 4	TASK 5	TASK 1	TASK 2	TASK 3	TASK 4	TASK 5
1	1.31	2.11	2.19	2.40	2.34	2.32	2.31	2.36	2.41	2.27
2	1.57	1.76	1.76	1.67	1.73	1.44	1.58	1.51	1.62	1.63
3	1.41	2.02	2.05	1.98	2.24	1.73	1.78	1.93	1.93	1.99
4	1.77	2.02	2.03	1.90	2.05	1.70	2.04	1.99	2.10	2.15
5	1.89	1.89	2.29	2.23	2.45	2.09	2.25	2.31	2.40	2.40
6	1.47	1.80	1.95	2.03	2.05	1.57	1.77	1.88	1.92	2.03
7	2.28	3.24	3.31	2.59	3.07	3.11	3.05	3.10	3.03	3.23
8	1.71	1.79	1.99	1.95	1.95	1.51	1.67	1.69	1.72	1.77
9	1.75	2.36	2.16	2.27	2.23	1.85	2.00	1.90	2.08	2.12
10	2.44	2.36	2.57	2.63	2.73	2.53	2.68	2.61	2.82	2.64
11	1.93	2.27	2.41	1.96	2.01	1.80	2.00	1.88	1.78	1.81
12	2.00	2.03	1.97	1.81	1.90	1.81	2.15	2.15	2.15	2.13
13	1.92	3.16	3.07	2.20	2.40	2.44	2.72	2.55	2.32	2.65
14	1.26	1.96	1.58	1.53	1.77	1.51	1.68	1.63	1.78	1.89
15	2.33	2.74	2.85	2.73	2.91	2.33	2.39	2.40	2.41	2.24
16	1.76	2.03	1.74	1.39	1.63					
17	1.59	2.30	1.72	1.34	1.64					
18	1.65	1.69	1.74	1.65	1.73					
19	2.06	2.59	2.70	1.85	1.87					
20	1.89	2.01	2.11	2.19	2.21					
21	1.55	2.08	1.89	2.05	2.22					
22	2.41	2.73	2.64	2.56	2.73					
23	2.13	3.01	2.25	2.15	2.36					
24	1.70	2.28	2.21	1.72	1.94					
25	2.86	3.00	2.86	2.36	2.83					
26	1.43	1.71	1.69	1.53	1.62					
27	1.96	1.79	1.91	1.91	2.26					
28	1.62	1.96	1.82	1.95	2.17					
29	1.63	2.70	1.84	1.82	1.79					
30	1.70	1.92	1.97	2.03	2.25					
31	1.67	2.01	1.90	1.91	2.01					
32	2.01	3.63	2.56	1.35	2.38					
33	1.73	1.98	2.10	1.63	1.91					

The reliabilities of the separate estimates of spectral density cannot be computed directly, since the frequency, p, is a relevant variable (i.e., there are eight U_p values for each subject and task, one for each of the eight values of p).

However, for every given subject, task, and value of p, the significance criterion (4.6) may be used to compare $U_{(test)}$ with $U_{(retest)}$. Under the null hypothesis that the time-series population of the observations in the jabbing experiment for a given subject and task is identical from test to retest, the ratio $\frac{U_{p\,(retest)}}{U_{p\,(test)}}$ is distributed as F with 9 and 9 degrees of freedom for p = 0, 1, 2, 3, 4, and with 27 and 27 degrees of freedom for p = 7, 12, 17. (See Appendix C, footnote 11.) The ratios $\frac{U_{p\,(retest)}}{U_{p\,(test)}}$ were computed for all 15 subjects, 5 tasks, and 8 frequencies, a total of 600 ratios. Table 4 indicates which of these ratios are significant at the 20% level of F (10% on each tail).[9] The total number of significant ratios is 198 out of 600, or 33.0%, which is considerably more than the expected 20% under the null hypothesis. A certain amount of unreliability in the spectra thus must be taken into consideration. As was the case with σ^2 and s, Task 1 again proves to be the least reliable of all the tasks, producing 49 of the 198 significant ratios.

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[9]The one-tailed 10% level of F for 9 and 9 d.f. is 2.44. For 27 and 27 d.f. it is 1.65. A loss of degrees of freedom is encountered when the spectrum departs considerably from flatness. For a discussion of this effect, the reader is referred to the memorandum by Tukey and Hamming (39). This effect is not of paramount importance here; on the basis of corrections for loss of degrees of freedom applied to a sampling of 59 of the 198 significant ratios, 4 of these become non-significant. With this correction, the percentage of significances would become approximately 31% instead of 33%.

TABLE 4

OCCURRENCES OF SIGNIFICANCES IN $\frac{U_{p(retest)}}{U_{p(test)}}$ FOR ALL SUBJECTS,

TASKS,* AND VALUES OF p

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}	TOTAL NO. OF SIGNIF'S
1	*245	2	2	1	15	2	5	135	13
2	45	5	5	12	1	123	1345	12345	19
3	12	12	1	12	12	-	-	1245	13
4	15	5	23	13	-	3	25	34	12
5	12	1	5	2	34	34	-	-	9
6	124	4	4	4	3	1234	125	-	14
7	-	-	1	1	135	5	25	35	10
8	123	123	-	-	-	2345	5	45	13
9	134	234	23	2	2	345	3	234	17
10	12	12	1	1	14	134	25	15	15
11	1245	125	1	2	-	235	34	3	15
12	234	4	-	5	35	5	45	1	11
13	1	1	1	13	1	14	34	-	10
14	15	15	1	1	-	45	124	345	14
15	24	24	24	34	-	234	24	-	13
TOTAL NO. OF SIG'S	34	24	16	19	17	33	27	28	198

*The numbers in each cell of the table denote the tasks for which significances occur for the particular subject and value of p. The total numbers of significances, task by task, are as follows: Task 1 - 49; Task 2 - 42; Task 3 - 34; Task 4 - 37; Task 5 - 36.

The general question of task differences was considered next.

From the composite spectra for all individuals, plotted in Figure 13, it is immediately apparent that Task 1 is quite different from the other tasks and that Tasks 2, 3, 4 and 5 are quite similar to each other. In addition, Task 1 differs from the others on both σ^2 and s. To test the homogeneity of Tasks 2, 3, 4 and 5 with reference to the measures σ^2 and s, an analysis of variance was performed on s and on <u>log</u> σ^2. (The logarithm is used to normalize the distribution.) These analyses used only those subjects (number 1 - number 15) who were retested, so that an estimate of the between-replications mean square would be available; in both analyses Task 1 was excluded from consideration. The results of these analyses are presented in Tables 5 and 6. It is seen that the components of variance due to tasks are considerably smaller than the components due to subjects in both cases. Furthermore, and perhaps more important, the subject-task interaction components are negligible. Accordingly, it was felt that Tasks 2, 3, 4 and 5 were not producing differential information of sufficient magnitude. <u>All measures</u> (U_p, σ^2, s) <u>were summed over these four tasks for each individual</u>.

Task 1 was excluded from all further analysis. (It might be argued that since the reliability of s for Task 1 is not <u>significantly</u> lower than the reliabilities of s for the other tasks, and since the apparent unreliability of σ^2 on Task 1 is a function of the shape of the spectrum for that task, that it is unwarranted to dispose of Task 1 so summarily. However, there are other reasons for its abandonment. The preponderance of low frequencies in the spectrum of Task 1 suggests that sudden shifts in the mean observation may be present in the data. Such shifts might

FIGURE 13. SPECTRA SUMMED OVER SUBJECTS NOS. 1-15
(RETEST)

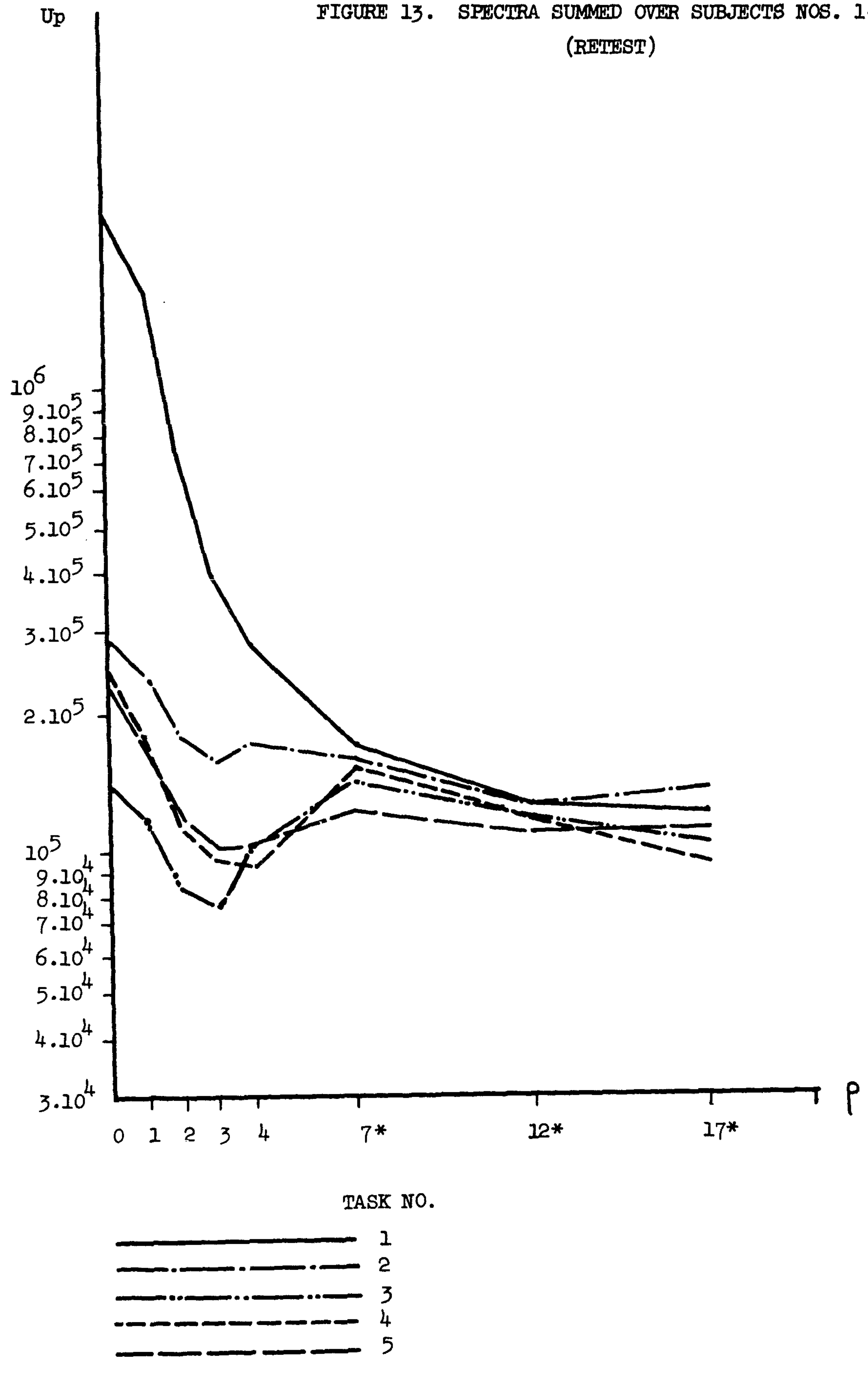

FIGURE 13. SPECTRA SUMMED OVER SUBJECTS NOS. 1-15
(TEST)

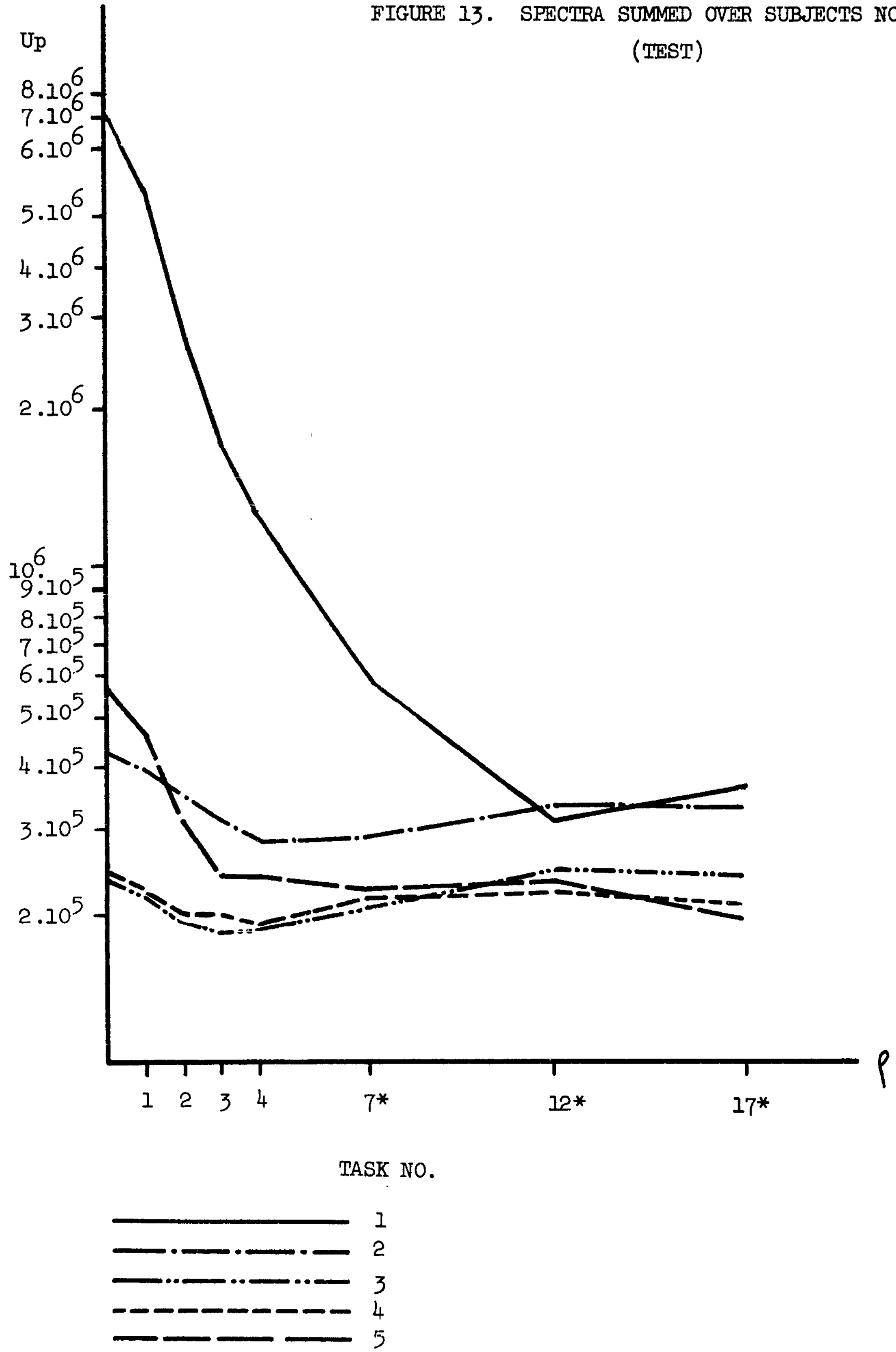

TABLE 5

ANALYSIS OF VARIANCE OF s FOR TASKS 2, 3, 4, 5

SOURCE	SUM OF SQUARES	D.F.	MEAN SQUARE	ESTIMATED VARIANCE COMPONENT
Total	20.6200	119		
Between Subjects	17.4147	14	1.2439	.145
Between Tasks	.1050	3	.0350	(-.001)
Between Replications	.1358	1	.1358	.001
Subjects X Tasks	1.2762	42	.0304	.008
Subjects X Replications	.9567	14	.0683	.013
Tasks X Replications	.1467	3	.0489	.002
Triple Interaction (Error)	.5849	42	.0139	.014

TABLE 6

ANALYSIS OF VARIANCE OF LOG σ^2 FOR TASKS 2, 3, 4, 5

SOURCE	SUM OF SQUARES	D.F.	MEAN SQUARE	ESTIMATED VARIANCE COMPONENT
Total	692.37	119		
Between Subjects	400.87	14	28.63	2.81
Between Tasks	54.69	3	18.23	.29
Between Replications	4.04	1	4.04	(-.16)
Subjects X Tasks	70.81	42	1.69	(-.02)
Subjects X Replications	86.46	14	6.18	1.11
Tasks X Replications	2.84	3	9.47	.52
Triple Interaction (Error)	72.66	42	1.73	1.73

ESTIMATED VARIANCE COMPONENTS (OF LOG σ^2)

WHEN THE ANALYSIS IS PERFORMED ON EACH TASK SEPARATELY

	Task 1	Task 2	Task 3	Task 4	Task 5
Subjects	6.45	2.21	3.02	3.79	2.14
Replications	4.66	(-.15)	.02	.07	(-.24)
Subj. X Repl. (Error)	10.18	2.29	2.34	2.96	3.78

be transients presenting non-stationarity; this would weaken the validity of spectral analysis. An inspection of the original data reveals a profusion of such shifts; it is difficult to specify the exact extent of the effects of such shifts, but at any rate Task 1 is open to suspicion on the grounds of possible non-stationarity. Further, the placement of Task 1 at the beginning of the experimental series makes it vulnerable to uncontrolled variations depending upon the subjects' initial lack of adjustment to the experimental situation. Most of these objections could be overlooked if Task 1 revealed interesting patterns of results but, unfortunately, the impression is one of chaos rather than of regularity.)

The summation over Tasks 2, 3, 4 and 5 reduced the four measures for σ^2 and for s to one measure for each (Tables 7 and 8). With the spectral densities, however, the problem remained as to how to summarize the information provided by the eight different values of U_p. The spectra for test and retest (with the U_p's summed over Tasks 2, 3, 4 and 5) for each of the fifteen retested subjects are presented in Figure 14. Inspection of these graphs suggests that there is a fair degree of consistency in the over-all shape of the two spectra produced by each particular subject, although the entire set of 30 curves does not seem to embrace one particular shape or family of shapes. In order to summarize the consistent characteristic of the spectra for each individual subject, an index α was devised to represent the proportion of the total area under the spectrum which lay to the left of a particular value of p. It is true from (B.9) that the total area under the spectrum is proportional to σ^2. The index α thus expresses the percentage of this total variance σ^2 which is accounted for by component waves of frequency lower than some value.

FIGURE 14. SPECTRA SUMMED OVER - TASKS 2,3,4,5

KEY: TEST ———— RETEST —·—·—

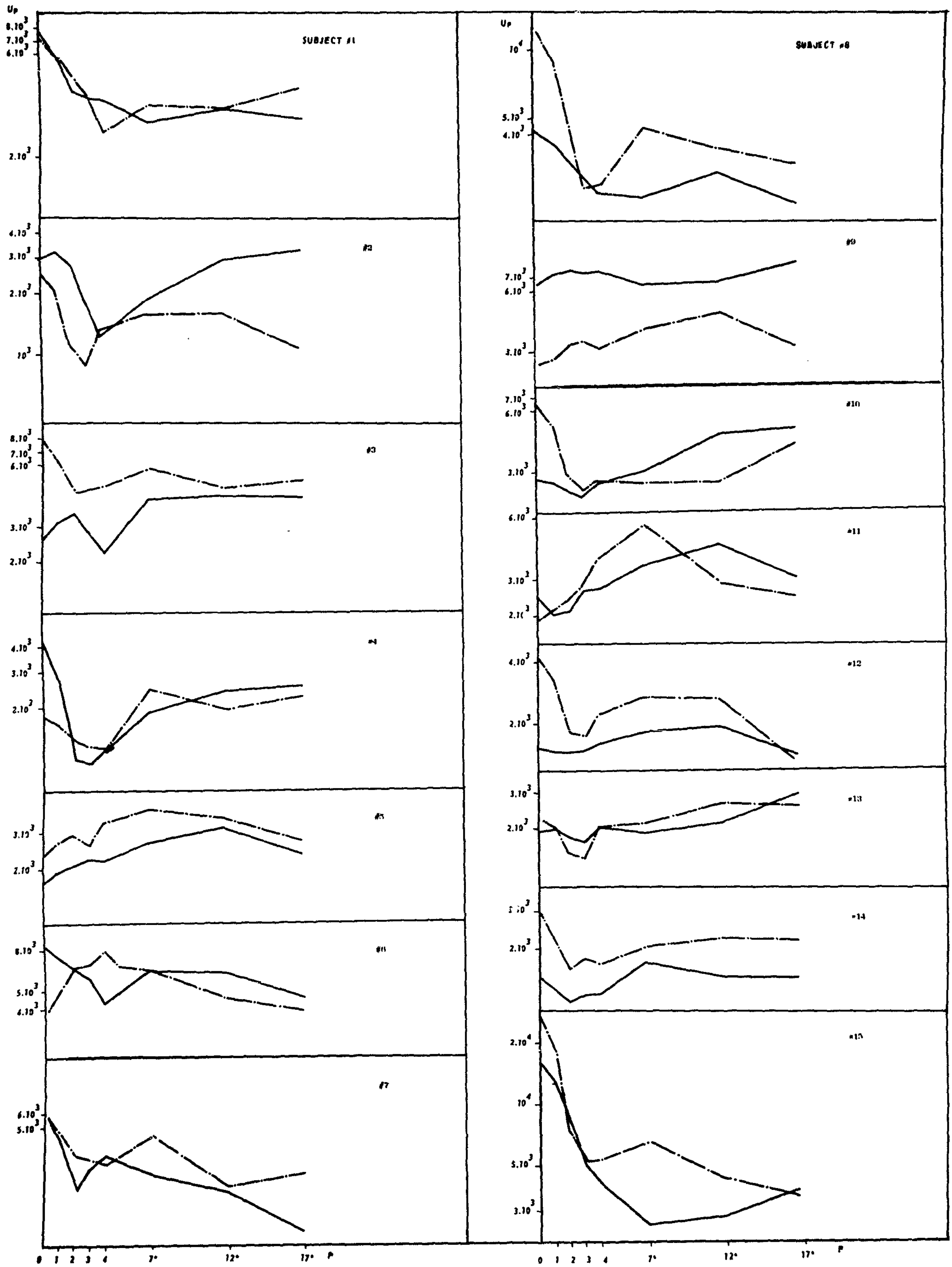

TABLE 7

VALUES OF σ^2 FOR ALL SUBJECTS SUMMED OVER TASKS 2, 3, 4, 5

SUBJECT	σ^2 (IN [FIFTIETHS-OF-AN-INCH]^2)	RETEST
1	178.1	205.4
2	132.3	72.0
3	200.7	267.6
4	111.1	105.0
5	132.6	169.0
6	301.3	264.4
7	137.4	176.4
8	114.7	194.8
9	357.5	187.1
10	198.9	167.3
11	176.0	179.4
12	82.1	115.1
13	115.0	120.9
14	72.7	107.6
15	211.5	309.9
16	127.1	
17	243.0	
18	100.7	
19	162.6	
20	131.4	
21	235.8	
22	119.8	
23	125.6	
24	92.0	
25	93.9	
26	176.9	
27	147.1	
28	154.2	
29	76.0	
30	177.1	
31	133.9	
32	199.7	
33	109.6	

TABLE 8

VALUES OF s FOR ALL SUBJECTS SUMMED OVER TASKS 2, 3, 4, 5

SUBJECT	s (IN JABS/SECOND)	RETEST
1	9.04	9.35
2	6.92	6.34
3	8.29	7.63
4	8.00	8.28
5	8.86	9.36
6	7.83	7.60
7	12.21	12.41
8	7.67	6.85
9	9.03	8.10
10	10.29	10.75
11	8.66	7.47
12	7.72	8.58
13	10.82	10.24
14	6.84	6.98
15	11.23	9.44
16	6.79	
17	6.99	
18	6.81	
19	9.00	
20	8.52	
21	8.24	
22	10.65	
23	9.77	
24	8.14	
25	11.05	
26	6.55	
27	7.87	
28	7.91	
29	8.15	
30	8.17	
31	7.83	
32	9.91	
33	7.62	

The value chosen (arbitrarily) was $p = 3\frac{1}{2}$, or 3.5 cycles every 40 observation points. The total area under the spectrum lying to the left of $p = 3\frac{1}{2}$ can be approximated by $\frac{1}{2} U_0 + U_1 + U_2 + U_3$. Thus

$$(7.1) \qquad \alpha = \frac{\frac{1}{2}U_0 + U_1 + U_2 + U_3}{\sigma^2} .$$

The values of α are presented in Table 9.

α, σ^2, and s are the three final measures of individual difference. α is based on spectral analysis, while σ^2 and s are not. The test-retest reliabilities of the three measures (as computed by product-moment correlation) are:

α	.852
σ^2	.600
s	.896

Using only the measures for the test, and discarding the retest, the values of these three indices for all thirty-three subjects were reduced to standard scores (Table 10). It is these standard scores which were employed in the validating procedure (Section VIII). The intercorrelations among the three indices are:

$$r_{\alpha s} = .304$$
$$r_{\alpha\sigma^2} = .075$$
$$r_{s\sigma^2} = .075$$

Further Results

The contention that task differences were unimportant requires further explanation, particularly in connection with the estimates of spectral density. Suppose that for each subject, replication, and

TABLE 9

VALUES OF α (PER CENT OF LOW FREQUENCIES)

BASED ON SPECTRA SUMMED OVER TASKS 2, 3, 4, 5

SUBJECT	TEST	(RETEST)
1	25.6	22.7
2	17.6	18.5
3	13.7	18.4
4	15.9	12.6
5	13.7	14.4
6	20.1	19.4
7	24.3	21.1
8	24.0	27.5
9	17.5	14.0
10	11.2	20.8
11	12.0	11.9
12	15.8	20.2
13	14.5	12.8
14	14.5	16.5
15	39.8	35.7
16	15.6	
17	12.8	
18	13.1	
19	14.3	
20	35.3	
21	38.7	
22	25.0	
23	14.6	
24	16.1	
25	24.6	
26	17.4	
27	14.7	
28	22.0	
29	35.0	
30	21.2	
31	13.6	
32	22.0	
33	15.3	

(α is expressed as a percentage.)

TABLE 10

STANDARD SCORES ON THREE MEASURES OF INDIVIDUAL DIFFERENCE

SUBJECT	INDEX: α	σ^2	s
1	.75	.36	.32
2	-.27	-.37	-1.17
3	-.79	.72	-.21
4	-.50	-.71	-.41
5	-.78	-.36	.19
6	.04	2.33	-.53
7	.60	-.29	2.55
8	.56	-.65	-.64
9	-.29	3.23	.31
10	-1.11	.69	1.19
11	-1.01	.33	.05
12	-.51	-1.17	-.61
13	-.69	-.64	1.57
14	-.69	-1.32	-1.23
15	2.61	.90	1.86
16	-.54	-.45	-1.26
17	-.91	1.40	-1.12
18	-.87	-.87	-1.25
19	-.71	.11	.29
20	2.03	-.38	-.05
21	2.47	1.28	-.24
22	.69	-.57	1.45
23	-.66	-.48	.83
24	-.47	-1.01	-.31
25	.63	-.98	1.73
26	-.31	.34	-1.43
27	-.66	-.13	-.50
28	.29	-.02	-.48
29	1.99	-1.27	-.31
30	.19	.35	-.29
31	-.80	-.34	-.53
32	.29	.71	.93
33	-.57	-.73	-.68

frequency, a comparison is made between the U-values for Tasks 2, 3, 4 and 5. This comparison is best made according to the significance of the ratio of <u>two</u> U-values; for this purpose, Task 4 is arbitrarily chosen as a standard against which to compare the other three tasks. The ratios $\frac{U_{p(\text{Task 2})}}{U_{p(\text{Task 4})}}$, $\frac{U_{p(\text{Task 3})}}{U_{p(\text{Task 4})}}$, $\frac{U_{p(\text{Task 5})}}{U_{p(\text{Task 4})}}$ were computed for all values of p, all subjects, and both test and retest. Note was taken of those ratios which were significantly greater or less than the 20% limits prescribed by the F-test of (4.6). Of 1152 such available ratios, 357 are significant (Table 11). This represents <u>31.0%</u>, as opposed to the expected 20%. Now remember that when the ratios $\frac{U_{p(\text{retest})}}{U_{p(\text{test})}}$ were tested for significance, <u>33.0%</u> of such "reliability ratios" were significant. In other words, the tasks are <u>no more different from each other than any one of them is different from itself when replicated</u>. Task differences are of the same order of magnitude as experimental error. It is thus fruitless to claim that the task has any effect on the performance. Further confirmation of task indifference is supplied by examining the <u>pattern</u> of significances in Table 11. These significances are distributed almost at random throughout the table. To illustrate this, consider the number of significances accumulated by subjects number 1 - number 15 on test and retest:

TABLE 11

OCCURRENCES OF SIGNIFICANCES IN $\frac{U_{p(\text{Task } 2,\ 3,\ \text{or } 5)}}{U_{p(\text{Task } 4)}}$ FOR ALL

SUBJECTS, TASKS,* AND VALUES OF p; RETEST

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}	TOTAL NO. OF SIGNIFICANCES
1	*23	2	-	35	35	3	-	3	9
2	35	35	-	-	-	3	-	23	7
3	2	2	2	2	2	-	-	5	6
4	-	-	-	-	-	5	5	23	4
5	-	-	-	-	-	-	-	-	0
6	-	-	25	25	235	3	3	2	10
7	25	25	2	2	-	-	-	235	9
8	2	2	2	-	-	-	-	5	4
9	235	235	-	-	-	2	-	235	10
10	235	235	25	2	23	3	5	35	15
11	-	-	-	3	-	5	-	25	4
12	3	3	-	-	3	5	3	2	6
13	25	2	-	-	-	235	-	2	7
14	5	-	-	-	-	-	23	-	3
15	23	23	235	235	3	235	235	3	18
TOTAL NO. OF SIG'S	20	17	10	11	10	14	9	21	112

*The numbers in each cell of the table denote the tasks for which significances occur for the particular subject and value of p.

TABLE 11

OCCURRENCES OF SIGNIFICANCES IN $\frac{U_{p(\text{Task } 2,\ 3,\ \text{or } 5)}}{U_{p(\text{Task } 4)}}$ FOR ALL

SUBJECTS, TASKS,* AND VALUES OF p; TEST

SUBJECT	U_0	U_1	U_2	U_3	U_4	U_{7*}	U_{12*}	U_{17*}	TOTAL NO. OF SIGNIFICANCES
1	*3	-	2	5	3	-	3	5	6
2	-	-	5	-	-	25	3	2	5
3	-	-	-	-	-	-	-	35	2
4	5	35	3	3	-	35	25	3	10
5	3	-	-	2	235	235	-	-	8
6	3	3	-	-	-	-	-	2	3
7	25	25	25	2	23	5	25	2	13
8	5	-	3	35	-	-	-	23	6
9	5	-	2	2	2	-	23	2	7
10	235	235	25	-	-	-	2	-	9
11	35	5	2	-	-	-	25	-	6
12	25	25	-	5	-	5	25	2	9
13	23	23	235	35	235	2	25	235	18
14	-	-	-	-	-	2	2	2	3
15	3	3	-	2	-	2	-	-	4
16	-	-	-	-	-	2	35	-	3
17	3	-	-	-	-	-	23	5	4
18	35	35	35	5	-	5	-	-	8
19	-	2	2	25	2	2	23	25	10
20	3	-	-	-	-	-	35	-	3
21	25	25	5	5	5	5	3	-	9
22	-	-	-	-	2	5	-	235	5
23	235	235	2	2	2	235	-	23	14
24	25	2	3	3	3	3	2	2	9
25	23	23	2	-	-	5	-	3	7
26	-	-	-	2	235	2	2	2	7
27	235	2	-	-	-	-	2	-	5
28	-	-	-	-	-	2	35	-	3
29	25	25	2	3	-	25	5	2	10
30	5	5	-	-	25	3	-	-	5
31	5	5	2	235	25	-	3	-	9
32	2	2	25	25	25	-	-	5	9
33	25	25	23	23	23	235	23	5	16
TOTAL NO. OF SIG'S	41	33	26	26	26	30	34	29	245

*The numbers in each cell of the table denote the tasks for which significances occur for the particular subject and value of p.

Number of significant ratios $\frac{U_p \text{ (Task 2, 3, or 5)}}{U_p \text{ (Task 4)}}$, all values of p

Subject No.	Test	Retest	Total
1	6	9	15
2	5	7	12
3	2	6	8
4	10	4	14
5	8	0	8
6	3	10	13
7	13	9	22
8	6	4	10
9	7	10	17
10	9	15	24
11	6	4	10
12	9	6	15
13	18	7	25
14	3	3	6
15	4	18	22
TOTAL	109	112	221

An analysis of variance of these numbers of significances according to subject and replication follows:

Source	Sum of Squares	D.F.	Mean Square
Total	548.97	29	
Replications	.30	1	.30
Subjects	262.47	14	18.74
Interaction	286.20	14	20.44

The between-subjects mean square is less than the interaction mean square; that is to say, there is a negative correlation between the number of significances accumulated by a subject on the test and the number accumulated on the retest. Thus it is not consistently true that some subjects respond alike to all tasks (producing few significances), while others respond differentially to them (producing many significances).

In assigning measures of individual difference to all thirty-three subjects, it has been tacitly assumed that the fifteen subjects who were retested are representative of the entire group of thirty-three. (These fifteen were chosen for a retest on a volunteer basis.) In other words, certain conclusions were based on the reliability demonstrated by the fifteen retested subjects. It is desirable to determine whether these fifteen are homogeneous with the non-retested eighteen on the three measures, α, σ^2, s. Accordingly, the mean and variance of the standard scores on the three measures for the retested group have been compared with the mean and variance of the standard scores for the non-retested group:

	Retested Group (N = 15)		Non-Retested Group (N = 18)	
	Mean	Variance	Mean	Variance
α	-.139	.858	.116	1.086
σ^2	.203	1.472	-.169	.542
s	.216	1.169	-.179	.787

The means are significantly different for none of the three measures. (If a t-test is used, all three values of t are less than 1.) The variances differ significantly only in the case of σ^2, where $F = 2.75$, $.05 > P > .01$. In general, we reach the conclusion that the group of fifteen retested subjects is an unbiased sample of the entire group of thirty-three subjects.

It has been stated in several places throughout this paper that learning and fatigue factors are outside the present concern. In particular, if either or both are present in considerable degree in the jabbing tasks, the time-series populations must be considered non-stationary, and the assumptions of spectral analysis are not strictly met. In what way would

learning and/or fatigue affect the time-series populations? Not in a change in the mean observation, certainly, since there is no more reason to expect a rise in the mean than a fall in the mean; these two changes are symmetrical and indifferent from each other. The variance, σ^2, of the observations would be expected to change, however. Learning (that is to say, increased skill in the motor activity) should decrease the variance, and fatigue (either motor or central) should increase the variance. To test the presence of these effects, the series of 100 observations for each particular task and (retested) subject was divided into halves: the first 50 observations vs. the last 50 observations. A table of the variances of the observations within each half appears as Table 12. There is an apparent tendency for the variances to decrease on the test and increase on the retest. A four-way analysis of variance (of log σ^2) performed on Table 12 yields the following non-negligible estimated variance components: Subjects, 5.15; Tasks, 1.11; Halves-Replications Interaction, .52; Subjects-Replications Interaction, 2.82; Subjects-Tasks-Replications Interaction, 1.27; Error, 3.04. No other component exceeds .30. The Halves-Replications Interaction <u>Mean Square</u> is significantly greater than the Error Mean Square at the 1% level. ($F = 10.62$ with 1 and 42 degrees of freedom.) But the Halves-Replications Interaction is a less important factor than even task differences, which are in turn much less important than subject differences. It is plausible to suppose that the downward trend in the variances on the test is an accommodation or learning effect, while the upward trend in the variances on the retest is a boredom effect. At any rate, these trends are a group phenomenon. No consistent individual differences in the tendency toward changing variances can be found. (The Subjects-Halves Interaction vari-

TABLE 12

VARIANCE WITHIN FIRST 50 OBSERVATIONS AND WITHIN LAST 50 OBSERVATIONS

SUBJECT	TASK 2		TASK 3		TASK 4		TASK 5	
	TEST	RETEST	TEST	RETEST	TEST	RETEST	TEST	RETEST
1a*	47.4	45.1	41.9	33.4	45.7	42.1	60.6	49.4
1b*	27.5	93.4	32.4	51.2	49.2	54.5	50.2	38.4
2a	41.1	25.8	33.5	20.3	30.3	17.2	51.3	18.9
2b	34.4	21.1	29.2	16.0	24.8	15.2	33.1	19.7
3a	69.3	94.9	63.1	43.9	36.4	53.1	56.5	37.4
3b	55.8	88.2	33.1	67.0	46.9	65.5	57.8	61.6
4a	35.5	25.3	19.0	32.3	32.5	26.3	21.2	16.3
4b	31.3	30.1	20.5	37.6	22.1	23.0	32.7	19.9
5a	26.4	51.9	38.3	42.2	27.1	27.6	28.5	46.1
5b	45.9	30.8	30.0	46.5	22.2	48.0	44.8	42.8
6a	121.9	53.0	75.4	81.2	66.2	43.9	64.1	74.3
6b	79.7	90.2	54.6	92.4	71.8	53.7	69.0	46.4
7a	74.2	42.9	33.0	36.6	22.4	31.0	25.4	57.1
7b	44.1	59.5	22.5	35.4	30.4	28.3	16.8	71.7
8a	39.3	72.3	28.6	21.0	28.4	31.2	19.0	42.0
8b	36.9	52.8	15.4	51.5	23.4	42.5	27.3	64.4
9a	150.2	68.6	99.5	40.2	64.8	25.8	52.9	33.5
9b	122.0	80.5	97.7	29.2	81.1	43.6	53.7	57.7
10a	52.3	70.9	51.8	46.5	58.1	44.2	60.9	23.7
10b	57.5	32.7	46.3	42.9	32.7	37.3	32.8	25.4
11a	44.8	47.4	48.8	32.4	52.4	41.9	37.1	41.2
11b	37.5	71.2	37.4	52.2	44.8	48.1	34.1	33.2
12a	20.8	35.6	26.2	21.2	17.9	35.4	17.6	27.6
12b	46.4	34.3	18.0	20.6	12.9	26.7	9.9	26.3
13a	47.3	41.6	25.6	33.8	17.1	27.9	22.2	23.8
13b	45.7	35.3	28.7	28.0	13.5	29.2	33.7	24.1
14a	33.4	35.4	14.9	17.7	14.2	27.6	13.2	31.6
14b	29.0	35.3	12.1	20.2	11.4	25.4	16.9	28.5
15a	43.7	44.3	53.0	34.0	60.5	68.5	54.1	39.5
15b	52.2	47.0	35.4	50.2	40.0	174.8	45.4	67.2

*a denotes first 50 observations
b denotes last 50 observations

ance component is negligible.) The trends in variance probably do not seriously impair the validity of the measures α, σ^2, and s. Though statistically significant, they are not numerically large. A small degree of non-stationarity can be tolerated by spectral analysis, so that α is little affected. We might expect, however, that σ^2 would be rendered somewhat unstable, and, indeed, this expectation is consistent with the low reliability of σ^2. It is unlikely that trends in the variance have any direct bearing on the speed of jabbing, s.

Remarks

It is disappointing that task differences did not appear. The reason for this is probably that the tasks are ambiguously stated and the subjects restructure all the task situations into a constant pattern to which they can conform. In all the jabbing tasks, the subject realizes that the best he can do is to jab generally within a certain band, and this is exactly what he tries to do, regardless of the limits prescribed by the specific target. The experimental situation is not stressful enough to force the subject to attempt to narrow his "tolerance band." The entire question of the effect of stress is left open for future experimentation.

The absence of between-task differences is fortunate in one respect. Due to the task similarity, the individual differences stand out in bolder relief. That the measures σ^2 and s appear as reliable indices of individual difference on routine, repetitive tasks is not surprising; they have turned up in the psychological literature before (16, 34; 8, 21). The measure α is more interesting because it is new; it derives solely from the spectrum. Perhaps α is not the best index with which to summarize the information provided by the spectrum; for one thing, its defi-

nition rests on an entirely arbitrary decision (the cut-off point at $p = 3\frac{1}{2}$). Ideally, a general curve-fit to the spectrum (as was done with Day's data) would be preferable. At any rate, α is surprisingly reliable. It remains to be seen whether it is valid.

<u>Summary</u>

Individual differences pervade the performance of all five of the experimental tasks. The task differences are surprisingly slight. Task 1 was discarded because of unreliability and danger of non-stationarity, and the results on the other four tasks were summed together. Three reliable measures of individual difference have been discovered. They are:

1. Differences in the proportion, α, of low-frequency components in the spectrum of the observations.

2. Differences in the variance, σ^2, of the observations about the mean observation.

3. Difference in the speed, s, of jabbing. The test-retest reliabilities of these measures are .85, .60 and .90 respectively. The measures are essentially uncorrelated with one another. The validation of these measures is presented in the following section.

VIII

VALIDATION OF THE MEASURES OF INDIVIDUAL DIFFERENCE

The expectation of Section VI was that individual differences in the proportion of low frequency components in the spectrum were indicative of individual differences in attention, involvement, and concern over errors; individual differences in the variance of the observations were expected to be indicative of individual differences in stability, level of aspiration, and possibly motor skill. There was no expectation concerning individual differences in the speed of jabbing. The factor of rigidity was expected to influence Task 1 alone.

The three reliable measures of individual difference discovered in the analysis of the experimental data and discussed in Section VII are: φ, the proportion of low frequency components in the spectrum; σ^2, the variance of the observations; and s, the speed of jabbing. Task 1 proved unsatisfactory on several counts and was abandoned, and Tasks 2, 3, 4 and 5, which were intended to provide a graded series of increasing stressfulness, proved to be insignificantly different from one another.

The independent personality data available on the thirty-three experimental subjects consisted mainly of a series of extensive personal interviews with the subjects by Mr. Roy Heath of the Study of Education at Princeton Project. Since Mr. Heath had an intimate knowledge of the subjects, and the author had only a general idea of the personality variables with which the experimental measures of individual difference might be correlated (see above), the following validating procedure was decided upon:

The author supplied to Mr. Heath a list of the names and standard scores of all subjects with standard scores above +1.00 or below -1.00 on each of the three measures α, σ^2, and s. (Due to skewness in the distribution of α, it was necessary to include cases with scores below -.80.) The measures were noncommittally entitled Factor I, Factor II, and Factor III, respectively. Mr. Heath was asked to induce the nature of such <u>personality</u> factors as might have produced the given divisions of the subjects into two extreme groups. With no further instruction (and with no knowledge of what the experimental measures were) he was able to do this for Factor III (s). When Factors I (α) and II (σ^2) proved difficult, a list of "clues" was provided. He was asked to consider the following list of words: "Motivation, persistence, consistency, action, attention, concern over errors, concentration, motor skill" in conjunction with the list of names of the high and low scorers and give free play to any ideas arising from consideration of the key words in order to find personality characteristics differentiating the high and low scorers. With his task thus made more specific, Mr. Heath was able to induce the nature of a personality factor to correlate with the author's Factor I, and two, to correlate with the author's Factor II. His statement of the three factors follows.

Factor I (α):

"This factor would be named <u>persistence of focus</u>. Like Factor III (see below) it is a persistence factor, but basically different in that Factor III refers to long-term persistence, i.e., the tendency to persevere in a task lasting beyond one minute in length. Factor I is more an "attention" factor, i.e., the ability to focus the attention on the perceptual

scene without distraction. Those in the minus group [Note: Those with a preponderance of high frequency components in the spectrum] are most likely to hold a focus for 10 to 20 seconds without distraction from either an internal or external source. The plus group [Note: Those with a preponderance of low frequency components in the spectrum] are more scattered; because of distraction or dissociative tendencies they would be apt to miss something (be off focus) in split seconds during the task. Some are probably distracted by internal sources while others are distracted from external sources. What matters with the plus group is the "state of mind" that exists at the time of testing. The internally distractable are more liable to distraction at some times than at others."

Factor II (σ^2):

"This factor appeared less palpable than the others. In general the plus group [Note: High variance] are more rigid in their approach to a task, more prone to persist in one mental set without considering other ways of looking at the task. The minus group [Note: Low variance] are more flexible, more imaginative in their approach to a task.

"Another possible factor is manual dexterity. In general, the minus group [Note: Low variance] are quite agile with their hands."

Factor III (s):

"This factor seemed intuitively to be a meaningful psychological factor. Basically, it should be labeled emotional commitment vs. non-commitment. The plus group [Note: High speed] , in contrast to the minus group [Note: Low speed], are cautious [!] , wary of extending the self, and therefore often behave in a desultory, casual, dabbling manner, often appearing less active and not inwardly caring about a task. The minus

group are more deeply motivated, and therefore are more active, persistent, intense, and involved in the task. It might be mentioned in passing that the minus group, although all usually well motivated in task performances, vary considerably in the nature of their motivation; e.g., some strive for status whereas for others the task itself may be intrinsically interesting."

To check the validity of these induced factors, Mr. Heath was asked to rate on a ten-point scale all thirty-three subjects (including the extremes) with regard to his four personality factors (two possibilities had been suggested for Factor II). These ratings appear in Table 13. The ratings of all the <u>unextreme</u> subjects (i.e., those whose scores had not been used in the induction of the personality factors) were then correlated with the standard scores (Table 10) on the corresponding experimental measures. Of course this correlation procedure is subject to severe attenuation, due to the omission of the extreme cases. The validity coefficients are as follows (due to the definitions of high and low, a negative correlation is always in the expected direction).

<u>Validity of Measures, Omitting Extreme Cases</u>

Experimental Measure	Personality Factor	n	r
q	Persistence of Focus	24	+.010
σ^2	(Manual Dexterity	25	-.432
	(Flexibility	25	-.163
s	Emotional Commitment	21	-.193

Of these, only the correlation between σ^2 and the manual dexterity rating is significant. The validities of q and s are of course disappointing. It is not known how unreliable the rankings might be over the severely

TABLE 13

RATINGS ON FOUR PERSONALITY VARIABLES

(10 is the highest rating; 1 is the lowest)

SUBJECT	I	II_1	II_2	III
1	9	5	7	4
2	6	7	4	4
3	6	5	4	5
4	7	5	7	4
5	5	5	8	8
6	8	2	4	9
7	4	8	4	2
8	8	5	5	4
9	7	4	4	3
10	3	4	5	2
11	8	5	6	6
12	4	7	6	4
13	8	8	7	8
14	2	8	7	9
15	2	4	2	2
16	9	8	4	5
17	8	5	4	6
18	9	7	10	8
19	7	5	4	8
20	3	6	7	8
21	2	6	5	4
22	7	8	5	6
23	8	5	3	4
24	5	3	2	3
25	4	8	3	7
26	8	7	7	7
27	7	6	4	8
28	5	8	8	9
29	1	9	7	7
30	4	9	5	6
31	6	5	4	4
32	4	6	8	4
33	3	9	8	6

Factor I $\equiv$ Degree of persistence of focus
Factor II_1 $\equiv$ Manual dexterity
Factor II_2 $\equiv$ Degree of flexibility
Factor III $\equiv$ Degree of emotional commitment to situations

limited range of the neutral (unextreme) subjects. As a further step, the extreme cases were included in the analysis and the validities recomputed (although the personality ratings of the extreme subjects are potentially biased in the direction of conforming to expectation, since it was on the basis of the knowledge of these extreme subjects' scores that the personality variables were originally induced).

Validity of Measures, Including Extreme Cases

Experimental Measure	Personality Factor	n	r
α	Persistence of Focus	33	-.513
σ^2	(Manual Dexterity	33	-.540
	(Flexibility	33	-.262
s	Emotional Commitment	33	-.340

All of these correlations are significant at the 5% level in the expected direction, except for the correlation between σ^2 and flexibility rating.

Discussion

Although the validity coefficients for the three measures are subject to bias, the original inductions of the personality variables are not. Let us inquire into the rationale for the connection between the experimental measures and the respective personality variables.

That σ^2 should be indicative of manual dexterity is not surprising. It is hardly necessary to comment further, except for the observation that our hopes for a connection between σ^2 and some variable more strictly in the personality domain were not well realized. The postulated connection with rigidity is quite moot. At first blush, it would seem that the rigid individual--the individual who is unwilling to tolerate variety of

performance--would generate a consistent run of performances, that is, a low variance. This conception would contradict the postulated connection between rigidity and high variance. On the other hand, it can be argued that a flexible individual will experiment until he improves his performance, thus producing, in the end, a lower variance than the rigid individual. On this latter basis, one would predict that individuals with high variance on Task 1 would show lower variance on the other tasks than would individuals with low variance on Task 1. This is not borne out. The entire case for a correlation between rigidity and high variance in the present experimental situation is on very shaky ground.

The induced personality factor for s is somewhat paradoxical. The people who jab at a faster rate are adjudged to be more cautious and reluctant to commit themselves to the performance of a task. The people who perform slowly, meanwhile, are adjudged to be more venturesome and apt to commit themselves persistently to a task. The picture which emerges here is evidently that the slow performers are taking the task seriously, whereas the fast performers are anxious to get the job finished. If this be the case, then the measure s is essentially a fortuitous index lacking generality of application to routine, repetitive tasks. The explanation is nevertheless amusing, and the index is of possible interest in some situations.

The main interest centers on the index α. The expectation was that inattention, low motivation, and/or little concern over errors would lead to a greater preponderance of low frequency components in the spectrum (i.e., a high value of α). The first of these alternatives has been verified. (Apparently the tasks did not involve the subjects deeply

enough for motivation to be a factor.) Let us examine the manner in which this comes about. Whenever the subject is inattentive or distracted, he is apt to produce a deviant observation. Unless he immediately re-focuses his attention, a series of deviant observations may be produced, all on one side of the mean. A succession of such incidents will tend to produce a series of observations of slowly-wandering wave-like character, as in Figure 1C. Thus the spectrum will tend toward low-frequency components.

If the spectrum is indeed a sensitive indicator of the power of attention, spectral analysis may open up for re-examination a good deal of old data from attention studies. In addition, since continual alertness is of vital importance in so many industrial and military tasks, spectral analysis provides a way to test for those individuals who are alert and those who are not. A recommendation for further experimentation is as follows:

Set the subjects a repetitive task to perform. Vary experimentally the conditions of distractability, from an optimum working situation to one in which there is a maximum of distraction. Study the effects on the spectrum, especially with an eye toward a more precise specification of the exact shape of the spectrum under the various conditions. Of course, individual differences in the distractability of the subjects will have to be taken into consideration.

IX

SUMMARY

For the purpose of studying individual differences in certain characteristics of the performance of repetitive, routine tasks, several methods of analysis of time-ordered data were considered. Much previous psychological work with time-ordered data had been characterized by highly arbitrary decisions based upon inspection of the data. This mode of procedure was rejected, as was the more objective method of quality control. Autocorrelational analysis was judged to be suitable except for the undesirable sampling properties of autocorrelations; finally, spectral analysis was chosen, since it possesses as much generality as autocorrelational analysis, and in addition, the sampling properties of the measures derived from it are convenient and orderly. Spectral analysis decomposes the total oscillation of a time-series into oscillations of varying rates, specifying the relative contribution of each of the components of oscillation.

Spectral analysis as an analytical tool was applied to previous psychological data; in one case, to a study of mental blocking, and in another, to a study of serial patterns of response in an auditory discrimination. In both cases, spectral analysis led to a modification and clarification of the conclusions that had been reached by the previous authors. Spectral analysis was then applied to the experimental data of the present paper.

The experimental task was that of jabbing a stylus repeatedly at a target line or lines. The deviations of the jabs from a reference line were measured in serial order, and the series were then subjected to a spectral

analysis. Thirty-three subjects were tested on five variations of the main task. Fifteen of the subjects were retested a month later. Task variations proved unimportant, but reliable individual differences were found in three measures, two of which did not depend on spectral analysis and one of which did. These three measures were:

1. α, the proportion of low frequency components in the spectrum.
2. σ^2, the variance of the observations about the mean observation.
3. s, the speed of production of observations.

It was disappointing to find that spectral analysis yielded only one measure of individual difference, since it had been hoped that a general model for the particular time-series process would have emerged on the basis of spectral analysis.

The three measures were related to general personality characteristics, by means of an inductive process. An individual who had had considerable contact with the experimental subjects in interview situations over a period of two years was asked to suggest personality factors which might have produced the differences found on each of the three measures. Without knowledge of the nature of the experimental measures, he suggested the following corresponding personality factors:

I. Degree of persistence of perceptual focus (i.e., power of attention)

II. Manual dexterity

III. Degree of emotional commitment to task situations.

On the basis of a subsequent rating procedure, the three experimental measures were found to be significantly valid in terms of the three personality factors, although there is a possibility that a bias influenced

the rankings in a favorable direction. Factors I and II were deemed to be rationally related to the corresponding experimental measures. Indeed, Factor I conformed precisely to previous expectation with regard to the meaning of the measure α. Factor III was somewhat of an accident and a paradox, although a rationale was found to give the factor meaning in terms of the experiment.

Suggestions for future experimentation on individual differences in the performance of repetitive tasks include (1) an attempt to increase the power of spectral analysis by concentrating on the attention factor and trying to relate the spectral results more generally and more directly to the process of "attending"; (2) an experimental study of the effect of stress on performance.

APPENDIX A

SPECTRAL DECOMPOSITION OF VARIANCE

The variance of a time series population for any arbitrary, fixed value of t is defined as the average squared deviation of the potentially occurring values of X_t from the mean value of X_t. Defining ave. $\{X_t\}$ to be the average of X_t <u>over the population</u>, the variance is:

$$\sigma_X^2 = \text{ave.}\left\{X_t^2\right\} - \left[\text{ave.}\left\{X_t\right\}\right]^2 \quad . \tag{A.1}$$

For a stationary time-series, the variance is independent of the particular arbitrary value of t.

Consider the simple, single cosine-component population $\{X_t\} = a_1 \cos(w_1 t + \phi_1)$, where a_1 and w_1 are fixed, and ϕ_1 is uniformly (i.e., rectangularly) distributed over the interval 0 to 2π. (That is, $f(\phi_1)\, d\phi_1 = \frac{d\phi_1}{2\pi}$). It is this distribution of phase angles ϕ_1 that determines the distribution of potentially occurring values of X_t. Now

$$\text{ave.}\left\{X_t\right\} = \int_0^{2\pi} a_1 \cos(w_1 t + \phi_1)\, f(\phi_1) d\phi_1 = \int_0^{2\pi} a_1 \cos(w_1 t + \phi_1) \frac{d\phi_1}{2\pi}$$

$$= \frac{a_1}{2\pi} \sin(w_1 t + \phi_1) \Bigg|_{\phi_1 = 0}^{\phi_1 = 2\pi} = \frac{a_1}{2\pi} \left[\sin(w_1 t + 2\pi) - \sin w_1 t\right] = 0$$

$$\text{ave.}\left\{X_t\right\} = 0 \quad . \tag{A.2}$$

$$\text{ave.}\left\{X_t^2\right\} = \int_0^{2\pi} a_1^2 \cos^2(w_1t + \phi_1)\,\frac{d\phi_1}{2\pi} = \frac{a_1^2}{2\pi}\left[\tfrac{1}{2}\sin(w_1t + \phi_1)\cos(w_1t + \phi_1) + \tfrac{1}{2}(w_1t + \phi_1)\right]_{\phi_1 = 0}^{\phi_1 = 2\pi}$$

$$= \frac{a_1^2}{4\pi}\left[\sin(w_1t + 2\pi)\cos(w_1t + 2\pi) - \sin w_1t \cos w_1t + (w_1t + 2\pi) - w_1t\right]$$

$$= \tfrac{1}{2} a_1^2 \qquad (\text{since } \sin(w_1t + 2\pi) = \sin w_1t\ ; \quad \cos(w_1t + 2\pi) = \cos w_1t)$$

(A.3) $$\text{ave.}\left\{X_t^2\right\} = \tfrac{1}{2} a_1^2 \quad .$$

Thus, from (A.1)

(A.4) $$\sigma_X^2 = \tfrac{1}{2} a_1^2 \quad .$$

For a general cosine-component time-series population,

$$\left\{X_t\right\} = \sum_{i=1}^{\infty} a_i \cos(w_it + \phi_i) \qquad ,$$

as is given in (4.1) as the basis for spectral analysis, where all a_i and w_i are fixed, and the ϕ_i are independently and uniformly distributed over the interval 0 to 2π. (That is, $f(\phi_1,\phi_2 \ldots \phi_i \ldots)\, d\phi_1 d\phi_2 \ldots d\phi_i \ldots = \frac{d\phi_1}{2\pi}\,\frac{d\phi_2}{2\pi} \ldots\ldots \frac{d\phi_i}{2\pi} \ldots\ldots$)

$$(A.5) \qquad \text{ave.}\left\{X_t\right\} = \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \cdots\cdots \int_0^{2\pi} \frac{d\phi_i}{2\pi} \cdots\cdots \sum_{i=1}^{\infty} a_i \cos(w_i t + \phi_i)$$

$$= \sum_{i=1}^{\infty} a_i \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \cdots\cdots \int_0^{2\pi} \frac{d\phi_i}{2\pi} \cos(w_i t + \phi_i) \ldots = 0$$

$$(A.6) \qquad \text{ave.}\left\{X_t^2\right\} = \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \cdots\cdots \int_0^{2\pi} \frac{d\phi_i}{2\pi} \cdots\cdots \left[\sum_{i=1}^{\infty} a_i \cos(w_i t + \phi_i)\right]^2$$

The square of the summation will produce terms of the form $a_i a_j \cos(w_i t + \phi_i)$ $\cos(w_j t + \phi_j)$ and, where $i = j$, of the form $a_i^2 \cos^2(w_i t + \phi_i)$. The cross-product terms ($i \neq j$) will all integrate out to zero in the manner of (A.2) or (A.5), since the integrals over ϕ_i and ϕ_j are independent. The remaining non-zero terms of (A.6) yield

$$(A.7) \qquad \text{ave.}\left\{X_t^2\right\} = \sum_{i=1}^{\infty} a_i^2 \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \cdots\cdots \int_0^{2\pi} \frac{d\phi_i}{2\pi} \cos^2(w_i t + \phi_i)$$

$$\int_0^{2\pi} \frac{d\phi_{i+1}}{2\pi} \cdots\cdots$$

The integral over ϕ_i yields $\frac{1}{2}$, in the manner of (A.3). The integrals over each of the other ϕ_i are unity. Thus

$$(A.8) \qquad \text{ave.}\left\{X_t^2\right\} = \sum_{i=1}^{\infty} \frac{1}{2} a_i^2 \qquad .$$

From (A.1), (A.5), and (A.8),

(A.9) $$\sigma_X^2 = \sum_{i=1}^{\infty} \frac{1}{2} a_i^2 \quad .$$

This is the result (4.2) stated in Section IV.

APPENDIX B

ESTIMATION OF SPECTRAL DENSITIES

To devise a means for estimating spectral densities, it is necessary to relate the spectrum $s(w)$ to the lag covariances Q_j. In the notation of Appendix A, the lag covariances are expressed by

$$(B.1) \qquad Q_j = \text{ave.}\left\{X_t \, X_{t+j}\right\} \quad .$$

Due to stationarity, $\text{ave.}\left\{X_t \, X_{t+j}\right\} = \text{ave.}\left\{X_0 \, X_j\right\}$. Using the general cosine-component population $\left\{X_t\right\} = \sum_{i=1}^{\infty} a_i \cos(w_i t + \phi_i)$, the relation for Q_j becomes:

$$(B.2) \qquad Q_j = \text{ave.}\left\{X_0 \, X_j\right\} = \text{ave.}\left\{\left[\sum_{i=1}^{\infty} a_i \cos \phi_i\right]\left[\sum_{i=1}^{\infty} a_i \cos(jw_i + \phi_i)\right]\right\}$$

$$= \text{ave.}\left\{\sum_{i=1}^{\infty} \sum_{\lambda=1}^{\infty} a_i a_\lambda \cos \phi_i \cos(jw_\lambda + \phi_\lambda)\right\}$$

$$= \sum_{i=1}^{\infty} \sum_{\lambda=1}^{\infty} a_i a_\lambda \, \text{ave.}\left\{\cos \phi_i \cos(jw_\lambda + \phi_\lambda)\right\}$$

where we have simply replaced the product of summations by a double summation over i and a substitute index λ.

Consider a single term in the summation (B.2) with $i \neq \lambda$, remembering that the averaging process proceeds over the joint distribution of the ϕ_i. The ϕ_i have been assumed independently and rectangularly distributed.

$$\text{(B.3)} \qquad \text{ave.}\left\{\cos \phi_i \cos(jw_\lambda + \phi_\lambda)\right\} = \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \cdots \int_0^{2\pi} \frac{d\phi_i}{2\pi} \cos \phi_i$$

$$\cdots \int_0^{2\pi} \frac{d\phi_\lambda}{2\pi} \cos(jw_\lambda + \phi_\lambda) \cdots$$

$$= \int_0^{2\pi} \frac{d\phi_i}{2\pi} \cos \phi_i \int_0^{2\pi} \frac{d\phi_\lambda}{2\pi} \cos(jw_\lambda + \phi_\lambda) \quad .$$

The double integration is identical with the product of the single integrations, since they are independent. But $\int_0^{2\pi} \cos \phi_i \, d\phi_i = 0$. Thus

$$\text{(B.4)} \qquad \text{ave.}\left\{\cos \phi_i \cos(jw_\lambda + \phi_\lambda)\right\} = 0 \qquad \text{if } i \neq \lambda \quad .$$

On the other hand, if $i = \lambda$, then

$$\text{(B.5)} \qquad \text{ave.}\left\{\cos \phi_i \cos(jw_i + \phi_i)\right\} = \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \cdots$$

$$\int_0^{2\pi} \frac{d\phi_i}{2\pi} \cos \phi_i \cos(jw_i + \phi_i) \cdots$$

$$= \int_0^{2\pi} \frac{d\phi_i}{2\pi} \cos \phi_i \cos(jw_i + \phi_i)$$

$$= \int_0^{2\pi} \frac{d\phi_i}{2\pi} \cos \phi_i \left[\cos jw_i \cos \phi_i - \sin jw_i \sin \phi_i \right]$$

$$= \cos jw_i \int_0^{2\pi} \frac{d\phi_i}{2\pi} \cos^2 \phi_i - \sin jw_i$$

$$\int_0^{2\pi} \frac{d\phi_i}{2\pi} \sin \phi_i \cos \phi_i$$

The first integral is equal to $\frac{1}{2}$ and the second integral vanishes. Thus

$$\text{(B.6)} \qquad \text{ave.} \left\{ \cos \phi_i \cos(jw_i + \phi_i) \right\} = \tfrac{1}{2} \cos jw_i \quad .$$

Substituting (B.4) and (B.6) into the summation (B.2) yields

$$\text{(B.7)} \qquad Q_j = \sum_{i=1}^{\infty} \tfrac{1}{2}\, a_i^2 \cos jw_i \quad .$$

But, from (4.3), $\frac{1}{2}\, a_i^2$ is equal to the increase in $S(w)$ in the neighborhood of the frequency w_i.

$$\text{(B.8)} \qquad Q_j = \sum_{i=1}^{\infty} \Delta \left[S(w_i) \right] \cos jw_i \quad .$$

If we deal with the continuous instead of the discrete case, then the summation over i becomes an integral over w, $\Delta \left[S(w_i) \right]$ becomes $dS(w)$, or $s(w)dw$, and the formula for Q_j reads

$$(B.9) \qquad \boxed{Q_j = \int_0^{\pi} \cos\ jw\ s(w)\ dw} \quad .$$

This equation expresses a unique one-to-one relationship between the Q_j and $s(w)$. If one is known, the other is determined. Note here that if $s(w) = k$, a constant, then $Q_j = k \int_0^{\pi} \cos\ jw\ dw = 0$ for $j \neq 0$, confirming the statement in Section IV that the spectrum of a random series is flat. Note also that with $j = 0$, (B.9) reads $Q_0 = \int_0^{\pi} s(w)\ dw$, which is the proper expression that the variance is equal to the area under the spectrum.

Equation (B.9) provides a key to a method for computing estimates of spectral density at various values of the frequency. For a given time-series sample, estimates of Q_j can be calculated from (3.2). Consider a linear combination of the Q_j, namely $\sum_{j=0}^{m} b_j\ Q_j \cos\ jw_0$, where the b_j are a set of "magic numbers" to be explained below, w_0 is the frequency at which we wish to estimate the spectral density, and m is the total number of lag covariances which have been calculated. Using (B.9),

$$(B.10) \qquad \sum_{j=0}^{m} b_j\ Q_j \cos\ jw_0 = \sum_{j=0}^{m} b_j \cos\ jw_0 \int_0^{\pi} \cos\ jw\ S(w)dw$$

$$= \int_0^{\pi} S(w)dw \left[\sum_{j=0}^{m} b_j \cos\ jw_0 \cos\ jw \right]$$

Now the summation in the square brackets constitutes the first m terms of the Fourier expansion of some function of $(w-w_0)$, say $V(w-w_0)$. If the b_j are chosen properly, $V(w-w_0)$ can be forced into the form of a

high, square peak centered around $w=w_0$. That is,

$$(B.11)\qquad V(w-w_0) \approx \begin{cases} \text{constant, } c & w_0 - \delta < w < w_0 + \delta \quad (\delta \text{ small}) \\ 0 & \text{otherwise} . \end{cases}$$

With such a function,

$$(B.12)\qquad \sum_{j=0}^{m} b_j\, Q_j \cos jw_0 = \int_0^{\pi} V(w-w_0)\, S(w)dw \approx 2\delta c\, S(w_0) = k\, S(w_0)$$

yielding an immediate estimate of spectral density.

One possible selection of the b_j is

$$(B.13)\qquad b_j = \left(1 - \frac{1}{(m+1)}\right) \quad .$$

In this case,

$$(B.14)\qquad V(w-w_0) = C \left(\frac{\sin\left[(m+1)\ (w-w_0)\right]}{\sin\ (w-w_0)}\right)^2 \quad ,$$

which is a reasonably good approximation to a square peak centered about $w=w_0$ with[10] width $\frac{2\pi}{m}$ and only small ripples about zero outside of this range. The value of b_j given by (B.13) can be used to good advantage in computing spectral densities by (B.12).

---

[10] The wider the peak, the greater the "computational blurring" referred to in (4.7).

An even better set of b_j's exists, however. The derivation is too involved to enter into here; only the numerical values of these b_j are presented. (Professor Tukey very kindly supplied these to the author.) Table 14 gives the values of b_j for $j=0$ through 20. These may be used when exactly 20 lag covariances have been computed (i.e., when $m=20$). Using these tabulated b_j, define

$$(B.15) \qquad p = \frac{w_0 m}{\pi} \qquad \text{and}$$

$$(B.16) \qquad T_{jp} = b_j \cos jw_0 = b_j \cos \frac{jp\pi}{m} \qquad .$$

Then, from (B.12)

$$(B.17) \qquad \boxed{U_p = \sum_{j=0}^{20} Q_j \, T_{jp}}$$

is proportional to an estimate of the spectral density $s(\frac{p\pi}{20})$. The matrix T_{jp} is given in Table 15. The Q_j are computed from (3.2). If U_p is calculated for each of the values $p = 0, 1, 2, \ldots, 20$, estimates of spectral density are available at equally spaced intervals all along the frequency range from $w = 0$ to π. Formula (B.17) is seen to be a matrix multiplication. The row vector Q, consisting of the 21 entries $(Q_0 \; Q_1 \; \ldots \; Q_j \; \ldots \; Q_{20})$, multiplied by the matrix T_{jp}, yields the row vector U with the 21 entries $(U_0 \; U_1 \; \ldots \; U_p \; \ldots \; U_{20})$.

TABLE 14

VALUES OF b_j USED

j	b_j
0	3566.74
1	3481.16
2	3392.97
3	3250.37
4	3059.62
5	2800.13
6	2582.08
7	2296.42
8	2001.86
9	1708.78
10	1441.79
11	1147.60
12	917.14
13	712.37
14	535.84
15	376.25
16	287.50
17	180.33
18	104.79
19	54.69
20	27.61

TABLE 15

T-MATRIX, COLUMNS 0-10

j \ p:	0	1	2	3	4	5	6	7	8	9	10
0	1783	1783	1783	1783	1783	1783	1783	1783	1783	1783	1783
1	3481	3437	3311	3102	2816	2462	2046	1580	1076	0544	0000
2	3393	3227	2745	1994	1048	0000	-1048	-1994	-2745	-3227	-3393
3	3250	2896	1911	0508	-1004	-2298	-3091	-3210	-2630	-1476	0000
4	3060	2475	0945	-0945	-2475	-3060	-2475	-0945	0945	2475	3060
5	2800	1980	0000	-1980	-2800	-1980	0000	1980	2800	1980	0000
6	2582	1518	-0798	-2456	-2089	0000	2089	2456	0798	-1518	-2582
7	2296	1043	-1350	-2268	-0710	1624	2184	0359	-1858	-2046	0000
8	2002	0619	-1620	-1620	0619	2002	0619	-1620	-1620	0619	2002
9	1709	0267	-1625	-0776	1382	1208	-1004	-1523	0528	1688	0000
10	1442	0000	-1442	0000	1442	0000	-1442	0000	1442	0000	-1442
11	1148	-0179	-1091	0521	0928	-0811	-0675	1023	0355	-1133	0000
12	0917	-0283	-0742	0742	0283	-0917	0283	0742	-0742	-0283	0917
13	0712	-0323	-0419	0704	-0220	-0504	0678	-0111	-0576	0635	0000
14	0536	-0315	-0166	0510	-0433	0000	0433	-0510	0166	0315	-0536
15	0376	-0266	0000	0266	-0376	0266	0000	-0266	0376	-0266	0000
16	0288	-0233	0089	0089	-0233	0288	-0233	0089	0089	-0233	0288
17	0180	-0161	0106	-0028	-0056	0128	-0172	0178	-0146	0082	0000
18	0105	-0100	0085	-0062	0032	0000	-0032	0062	-0085	0100	-0105
19	0055	-0054	0052	-0049	0044	-0039	0032	-0025	0017	-0009	0000
20	0014	-0014	0014	-0014	0014	-0014	0014	-0014	0014	-0014	0014

TABLE 15

T-MATRIX, COLUMNS 11-20

j \ p:	11	12	13	14	15	16	17	18	19	20
0	1783	1783	1783	1783	1783	1783	1783	1783	1783	1783
1	-0544	-1076	-1580	-2046	-2462	-2816	-3102	-3311	-3437	-3481
2	-3227	-2745	-1994	-1048	0000	1048	1994	2745	3227	3393
3	1476	2630	3210	3091	2298	1004	-0508	-1911	-2896	-3250
4	2475	0945	-0945	-2475	-3060	-2475	-0945	0945	2475	3060
5	-1980	-2800	-1980	0000	1980	2800	1980	0000	-1980	-2800
6	-1518	0798	2456	2089	0000	-2089	-2456	-0798	1518	2582
7	2046	1858	-0359	-2184	-1624	0710	2268	1350	-1043	-2296
8	0619	-1620	-1620	0619	2002	0619	-1620	-1620	0619	2002
9	-1688	-0528	1523	1004	-1208	-1382	0776	1625	-0267	-1709
10	0000	1442	0000	-1442	0000	1442	0000	-1442	0000	1442
11	1133	-0355	-1023	0675	0811	-0928	-0521	1091	0179	-1148
12	-0283	-0742	0742	0283	-0917	0283	0742	-0742	-0283	0917
13	-0635	0576	0111	-0678	0504	0220	-0704	0419	0323	-0712
14	0315	0166	-0510	0433	0000	-0433	0510	-0166	-0315	0536
15	0266	-0376	0266	0000	-0266	0376	-0266	0000	0266	-0376
16	-0233	0089	0089	-0233	0288	-0233	0089	0089	-0233	0288
17	-0082	0146	-0178	0172	-0128	0056	0028	-0106	0161	-0180
18	0100	-0085	0062	-0032	0000	0032	-0062	0085	-0100	0105
19	0009	-0017	0025	-0032	0039	-0044	0049	-0052	0054	-0055
20	-0014	0014	-0014	0014	-0014	0014	-0014	0014	-0014	0014

APPENDIX C

COMPUTATIONAL PROCEDURE

The computations of spectral densities for the present experimental data and for the data of Section V were carried out on IBM punched-card machines. For each series of 100 observations (or for whatever number of observations was available in the case of the data of Section V), the first 20 lag covariances were computed. This gives high enough accuracy; more lags would not add enough information to justify the additional labor. Instead of computing U_p for all values of p from 0 to 20, only the values p = 0, 1, 2, 3, and 4 were computed separately. In addition, three more U-values representing averages were found, namely:

$$(C.1)\qquad \begin{aligned} U_{7*} &= \tfrac{1}{5}(U_5 + U_6 + U_7 + U_8 + U_9) \\ U_{12*} &= \tfrac{1}{5}(U_{10} + U_{11} + U_{12} + U_{13} + U_{14}) \\ U_{17*} &= \tfrac{1}{5}(U_{15} + U_{16} + U_{17} + U_{18} + U_{19}) \end{aligned}$$

Machine time was greatly reduced by the use of these averages, and the information loss was small, especially since the most interesting effects in the data showed up at the lower end of the spectrum.

The relevant formulas for the computation of the U_p are (3.2), (B.17), and (B.16).

These read (with m=20 and N=100)

$$(C.2)\qquad Q_j = \frac{\sum_{t=1}^{100-j} X_t X_{t+j}}{100-j} - \frac{\left[\sum_{t=1}^{100-j} X_t\right]\left[\sum_{t=1}^{100-j} X_{t+j}\right]}{(100-j)^2}$$

(C.3) $$U_p = \sum_{j=0}^{20} Q_j \, T_{jp}$$

(C.4) $$T_{jp} = b_j \cos \frac{jp}{20} \quad ,$$

where the numerical values of b_j are given in Table 13. The averaging involved in U_{7*}, U_{12*}, and U_{17*} was achieved by averaging columns of T_{jp}, thus:

$$U_{7*} = \frac{1}{5}(U_5 + U_6 + U_7 + U_8 + U_9)$$

$$= \frac{1}{5}\left(\sum_{j=0}^{20} Q_j T_{j5} + \sum_{j=0}^{20} Q_j T_{j6} + \sum_{j=0}^{20} Q_j T_{j7} + \sum_{j=0}^{20} Q_j T_{j8} + \sum_{j=0}^{20} Q_j T_{j9}\right)$$

$$= \sum_{j=0}^{20} Q_j \cdot \frac{1}{5}(T_{j5} + T_{j6} + T_{j7} + T_{j8} + T_{j9}) \quad .$$

Or,

(C.5) $$U_{7*} = \sum_{j=0}^{20} Q_j T_{j7*} \quad .$$

Similarly,

$$U_{12*} = \sum_{j=0}^{20} Q_j T_{j12*}$$

$$U_{17*} = \sum_{j=0}^{20} Q_j T_{j17*}$$

the star denoting an average taken over the indicated value and the two values on either side. If (C.2) is substituted into (C.3), we may write

$$U_p = \sum_{j=0}^{20} \frac{T_{jp}}{(100-j)^2} \left\{ (100-j) \sum_{t=1}^{100-j} X_t X_{t+j} - \sum_{t=1}^{100-j} X_t \sum_{t=1}^{100-j} X_{t+j} \right\}. \tag{C.5}$$

Written in this way, the formula is much more convenient for IBM purposes, since the factor of $(100-j)^2$ can be included beforehand in an adjusted T-matrix and the quantity in the brackets can be easily programmed for a calculating punch machine. The final computational procedure included one further simplification. Certain of the terms in $\Sigma X_t X_{t+j}$, ΣX_t, and ΣX_{t+j} of (C.6) were dropped in the interest of neat IBM procedures, as follows:

(C.7) For

j=0,	the terms	t = 1, 2, 3, 4, 5	were dropped
j=1,		t = 1, 2, 3, 4	" "
j=2,		t = 1, 2, 3	" "
j=3,		t = 1, 2	" "
j=4,		t = 1	" "
j=5,		none	" "
j=6,		t = 1, 2, 3, 4	" "
j=7,		t = 1, 2, 3	" "
j=8,		t = 1, 2	" "
j=9,		t = 1	" "
j=10,		none	" "
j=11,		t = 1, 2, 3, 4	" "
j=12,		t = 1, 2, 3	" "
j=13,		t = 1, 2	" "
j=14,		t = 1	" "
j=15,		none	" "
j=16,		t = 1, 2, 3, 4	" "
j=17,		t = 1, 2, 3	" "
j=18,		t = 1, 2	" "
j=19,		t = 1	" "
j=20,		none	" "

The effect of this was to retain

(C.8)

95 values of t for j = 0, 1, 2, 3, 4, 5
90 values of t for j = 6, 7, 8, 9, 10
85 values of t for j = 11, 12, 13, 14, 15
80 values of t for j = 16, 17, 18, 19, 20

The number of values of t retained for a given j will be denoted by N_j. The number of terms dropped constituted 2.4% of the total number of available terms.[11] (The schema for this abandonment of terms will be made clearer for the reader by reference to Figure 15.)

With all these modifications (averaging columns of T for p = 7, 12 and 17; factoring out $\frac{1}{(100-j)^2}$; dropping certain values of t) considered the final computational form can be written:

$$\text{(C.9)} \qquad \boxed{U_p = \sum_{j=0}^{20} T_{jp}\, C_j}$$

where T_{jp} is an element of the adjusted T-matrix, presented in Table 16, and

$$\text{(C.10)} \qquad C_j = N_j \sum_{t=\alpha}^{100-j} X_t X_{t+j} - \sum_{t=\alpha}^{100-j} X_t \sum_{t=\alpha}^{100-j} X_{t+j} \quad ,$$

where $\alpha = 100 - N_j - j + 1$ and N_j is given by (C.8).

---

[11] The dropping of terms serves to reduce the number of degrees of freedom available for the significance tests given in (4.5) and (4.6). With N = 100 observations and m = 20 lags, the degrees of freedom should be $\frac{2N}{m} = 10$. But in view of the dropped terms, 9 d.f. have been employed instead throughout this paper. The averaging of U-values in (C.1) also affects the degrees of freedom. Since neighboring U-values are not independent, but "neighboring-save-one" U-values <u>are</u> independent (see (4.7)), an average of 5 successive U-values deserves <u>3</u> times the number of degrees of freedom for each U-value; the two U-values on the ends of the string of five and the one in the middle each contribute degrees of freedom independently. Thus for p = 0, 1, 2, 3, 4, the d.f. are 9; for p = 7*, 12*, and 17*, the d.f. are 27.

TABLE 16

τ, THE ADJUSTED T-MATRIX

j \ p:	0	1	2	3	4	7*	12*	17*
0	1976	1976	1976	1976	1976	1976	1976	1976
1	3857	3810	3669	3437	3121	1708	-1163	-3353
2	3760	3576	3041	2210	1162	-1998	-2750	1998
3	3602	3209	2117	0563	-1113	-2816	2306	-0448
4	3390	2743	1048	-1048	-2743	-0678	0678	-0678
5	3103	2194	0000	-2194	-3103	1059	-1498	1059
6	3188	1874	-0985	-3032	-2579	0945	0307	-0945
7	2835	1287	-1666	-2800	-0876	0065	0336	0410
8	2471	0764	-1999	-1999	0764	0000	0000	0000
9	2110	0330	-2008	-0958	1707	0222	0077	-0113
10	1780	0000	-1780	0000	1780	0000	-0356	0000
11	1588	-0248	-1511	0721	1285	-0344	0119	0175
12	1269	-0392	-1027	1027	0392	-0254	0254	-0254
13	0986	-0448	-0580	0974	-0305	0033	-0173	0211
14	0742	-0436	-0229	0705	-0600	0112	-0036	-0112
15	0521	-0368	0000	0368	-0521	0031	0043	0031
16	0449	-0363	0139	0139	-0363	0000	0000	0000
17	0282	-0251	0166	-0044	-0087	0022	0018	0003
18	0164	-0156	0132	-0096	0051	0014	-0019	-0014
19	0085	-0084	0081	-0076	0069	-0007	-0005	0014
20	0022	-0022	0022	-0022	0022	-0004	0004	-0004

The IBM procedure proceeded in three main steps.

1. Calculating $\sum_{t=\alpha}^{100-j} X_t X_{t+j}$, $\sum_{t=\alpha}^{100-j} X_t$, and $\sum_{t=\alpha}^{100-j} X_{t+j}$

2. Calculating C_j by (C.10)

3. Calculating U_p by (C.9)

The discussion which follows pertains to the set of 100 observations for a single subject on a single task. It is understood, of course, that all 240 subject-task units underwent each computational step together before the next step was undertaken.

The card layout for step 1 is presented in Figure 15. To get the cards in this form required an "off-set gang punch." Only six master cards were originally punched, as follows:

Field:	34	33	32		22	21	20	19		1	0
Card No. 100	X_{66}	X_{67}	X_{68}	....	X_{78}	X_{79}	X_{80}	X_{81}	....	X_{99}	X_{100}
Card No. 86	X_{52}	X_{53}	X_{54}	....	X_{64}	X_{65}	(Blank)				
Card No. 72	X_{38}	X_{39}	X_{40}	....	X_{50}	X_{51}	(Blank)				
Card No. 58	X_{24}	X_{25}	X_{26}	....	X_{36}	X_{37}	(Blank)				
Card No. 44	X_{10}	X_{11}	X_{12}	....	X_{22}	X_{23}	(Blank)				
Card No. 34	C	X_1	X_2	..X_9	(Blank from field 24 to field 0)						

Between these master cards were collated blank cards, one for each missing card number (except numbers 1-5). The entire deck, in consecutive card number order from number 100 down to number 6 was passed through a reproducer wired to off-set gang punch. This operation punches the information in field (j+1), card (k+1) into field j, card k, for all j and k. It will be seen that the result of this step is to produce cards punched according

FIGURE 15. IBM CARD LAYOUT

(FIELDS 34-21)	F_{20}	F_{19}	F_{18}	F_{17}	F_{16}	F_{15}	F_{14}	F_{13}	F_{12}	F_{11}	F_{10}	F_9	F_8	F_7	F_6	F_5	F_4	F_3	F_2	F_1	F_0	(F = 2 COLUMN FIELD)
↓	X_{80}	X_{81}	X_{82}	X_{83}	X_{84}	X_{85}	X_{86}	X_{87}	X_{88}	X_{89}	X_{90}	X_{91}	X_{92}	X_{93}	X_{94}	X_{95}	X_{96}	X_{97}	X_{98}	X_{99}	X_{100}	CARD NO. 100
(used in	X_{79}	X_{80}	X_{81}	X_{82}	X_{83}	X_{84}	X_{85}	X_{86}	X_{87}	X_{88}	X_{89}	X_{90}	X_{91}	X_{92}	X_{93}	X_{94}	X_{95}	X_{96}	X_{97}	X_{98}	X_{99}	CARD NO. 99
preparation,	X_{78}	X_{79}	X_{80}	X_{81}	X_{82}	X_{83}	X_{84}	X_{85}	X_{86}	X_{87}	X_{88}	X_{89}	X_{90}	X_{91}	X_{92}	X_{93}	X_{94}	X_{95}	X_{96}	X_{97}	X_{98}	CARD NO. 98
but not for computation)	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}	X_{21}	X_{22}	CARD NO. 22
	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}	X_{21}	CARD NO. 21
	c	(X_1)	(X_2)	(X_3)	(X_4)	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	X_{20}	CARD NO. 20
	c	c	(X_1)	(X_2)	(X_3)	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	X_{19}	CARD NO. 19
	c	c	c	(X_1)	(X_2)	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	X_{18}	CARD NO. 18
	c	c	c	c	(X_1)	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	X_{17}	CARD NO. 17
	c	c	c	c	c	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	X_{16}	CARD NO. 16
c	c	c	c	c	c	c	(X_1)	(X_2)	(X_3)	(X_4)	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}	CARD NO. 15
indicates	c	c	c	c	c	c	c	(X_1)	(X_2)	(X_3)	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	X_{14}	CARD NO. 14
a control	c	c	c	c	c	c	c	c	(X_1)	(X_2)	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}	CARD NO. 13
punch	c	c	c	c	c	c	c	c	c	(X_1)	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}	CARD NO. 12
	c	c	c	c	c	c	c	c	c	c	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	CARD NO. 11
	c	c	c	c	c	c	c	c	c	c	c	(X_1)	(X_2)	(X_3)	(X_4)	X_5	X_6	X_7	X_8	X_9	X_{10}	CARD NO. 10
	c	c	c	c	c	c	c	c	c	c	c	c	(X_1)	(X_2)	(X_3)	X_4	X_5	X_6	X_7	X_8	X_9	CARD NO. 9
	c	c	c	c	c	c	c	c	c	c	c	c	c	(X_1)	(X_2)	X_3	X_4	X_5	X_6	X_7	X_8	CARD NO. 8
	c	c	c	c	c	c	c	c	c	c	c	c	c	c	(X_1)	X_2	X_3	X_4	X_5	X_6	X_7	CARD NO. 7
	c	c	c	c	c	c	c	c	c	c	c	c	c	c	c	X_1	X_2	X_3	X_4	X_5	X_6	CARD NO. 6
	(SECTION I)					(SECTION II)					(SECTION III)					(SECTION IV)						(CARDS NO. 1 - NO. 5 WERE DROPPED)

to the layout of Figure 15. The sums and sums of cross-products needed for (C.10) were computed by progressive digiting (20), using field 0 for the master sort. (Since the X's were two-digit numbers, the two alternatives for progressive digiting were a double sort or the preparation of a "reverse-digit" deck. The latter was used.) The cards were run through the tabulator twice, once for computing the progressive totals for all fields in Sections III and IV of Figure 15 vs. field 0, and again to treat all fields in Section I and II of Figure 15 vs. field 0. Any cards that included control punches within a particular section were automatically excluded from the totals <u>for that section</u>. A card count within each section was used to check the exclusion and to provide the values N_j in the summary cards.

The operation of progressive digiting produces one summary card for each digit of the master sort. The card corresponding to the zero digit contained the information $\sum_{t=\alpha}^{100-j} X_t$ and $\sum_{t=\alpha}^{100-j} X_{t+j}$. The cards corresponding to the remaining digits, when summed, produced the information $\sum_{t=\alpha}^{100-j}{}' X_t X_{t+j}$, in addition to N_j, which was carried along. Thus a pair of cards contained the four quantities necessary to compute the C_j from (C.10). (Step 2.) (Actually, since the progressive digiting was done in two runs through the tabulator there were two such pairs of cards, one pair with the information for $j = 0$ to 10, the other covering $j =$ 11 to 20.)

Step 2 was performed on a calculating punch. Behind each pair of cards was placed a trailer card on which was punched a single calculated

value of $C_j = N_j \sum_{t=\alpha}^{100-j} X_t X_{t+j} - \sum_{t=\alpha}^{100-j} X_t \sum_{t=\alpha}^{100-j} X_{t+j}$. Due to the limited storage capacity on the machine, 21 runs were necessary to calculate all 21 values of C_j. Each C_j was punched on a separate trailer card. Then, into the card with a particular C_j on it was punched the j:th row of the τ_{jp}-matrix; that is, the values $\tau_{j0}, \tau_{j1}, \tau_{j2}, \tau_{j3}, \tau_{j4}, \tau_{j7*}, \tau_{j12*}, \tau_{j17*}$.

Step 3 was also performed on the calculating punch. With 21 cards for each subject-task unit grouped together and followed by a trailer card the machine cumulated the sum of products $\sum_{j=0}^{20} C_j \tau_{jp}$ and punched the result into the trailer card. Eight runs through the machine were necessary, one for each value of p. The same trailer cards were retained throughout all eight runs, only the punching field changing. Thus one card was produced for each subject-task unit, and contained all eight U_p for that unit. <u>All</u> digits were retained in all computational steps up to the matrix multiplication. At that point, the last 8 places of the U_p were dropped, retaining from 2 to 5 places. No decimal points appear in the U_p, since an arbitrary multiplicative factor is implicit in all the U_p.

The computational checks carried at each of the three main steps were:

1. A comparison on field 0 (with a collator) between the standard and reverse-digit off-set gang-punched decks (which were produced independently). There was no check of the tabulator steps.

2. One out of every ten computations of C_j (Step 2) were spot-checked on a desk calculator. A complete re-run through the calculating

punch would have been preferable, although no errors were found during the spot-check.

3. For the matrix multiplication, the sum row over all subject-task units of the C_j was multiplied by the τ-matrix and checked against the sum row of the U_p over all units. That is, $\sum_{j=0}^{20} \tau_{jp} \sum_{\beta=1}^{240} C_j = \sum_{\beta=1}^{240} U_p$, where β denotes the unit.

In addition, eight units (of 100 observations each) for which the spectra were known were sent through the computational machinery and the results checked against the known values.

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ABSTRACT

For the purpose of studying individual differences in the performance of routine, repetitive tasks, mathematical methods of analysis of data which is ordered along the time dimension are considered. Spectral analysis is chosen as the best method, since it gives a complete summarization of the stationary properties of time-ordered data, and the measures derived from it possess convenient and orderly statistical properties. Spectral analysis decomposes the total oscillation of a time-series into oscillations of varying rates, specifying the relative contribution of each of the components of oscillation.

Spectral analysis as an analytical tool is applied to available psychological data; in one case, to a study of "mental blocking," and in another, to a study of serial patterns of response in an auditory discrimination. In both cases, spectral analysis leads to a modification and clarification that had been reached by the previous authors. Spectral analysis is then applied to the experimental data of the present paper.

The experimental task was that of jabbing a stylus repeatedly at a target line or lines. The deviations of the jabs from a reference line were measured in serial order, and the series were then subjected to a spectral analysis. Thirty-three subjects were tested on five variations of the main task. Fifteen of the subjects were retested a month later. Task variations proved unimportant, but reliable individual differences were found in three measures, two of which did not depend on spectral analysis and one of which did.

These three measures were related to general personality characteristics by means of an inductive process. An individual who had had considerable contact with the experimental subjects in interview situations over a period of two years was asked to suggest personality factors which might have produced the differences found on each of the three measures. Without knowledge of the nature of the experimental measures, he suggested personality factors corresponding to them. These personality factors and their relation to routine, repetitive tasks are discussed.